

The Tipping Point: Low Rates and Financial Stability^a

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Abstract

To study the effect on financial stability of persistent changes in the interest rate, this paper develops a recursive model of liquidity creation based on [Diamond and Dybvig \(1983\)](#). The model features two stable balanced growth paths: a good one with a healthy banking system and a bad one with a failed banking system. The paper's main result is that a critical interest-rate level exists, below which a financial crisis takes place and the economy transitions from the good to the bad BGP. At this tipping point, banks' franchise value of deposits decreases, since their net interest margins are compressed. This leads to a fall in bank equity, which gives depositors an incentive to run. The tipping point is not necessarily negative or zero. Importantly, it is an increasing function of the persistence of the change in the interest rate. Since a persistent fall in the interest rate compresses the net interest margin further in the future, it damages the franchise value of deposits more for any given interest-rate cut.

Keywords: Franchise value of deposits, liquidity, lower bound.

JEL Codes: E43, E50, G21.

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1 Introduction

Low interest rates by historical standards have prevailed in the developed world since the global financial crisis and are predicted to define the macroeconomic environment of the future (Kiley and Roberts, 2017). Thus, it is valuable to study the effects of low rates on other economic outcomes.

Banks are singled out as most prominent losers from a low-rate environment. It has been established empirically that banks are subject to a zero lower bound on the interest rates that they offer to retail depositors (Heider et al., 2018). Once this constraint is binding, a fall in the interest rate compresses banks' net interest margin and, as a consequence, their profitability. The adverse impact of low rates on bank profitability is likely to have major macroeconomic consequences. To study these, our view of the banking system's role in the economy is key.

This paper takes the view that banks are providers of liquidity. Banks offer demandable deposits to consumers. This is a good contract for the consumer, because he or she does not know at what point in time the need to consume may arise. It is also a good contract for the bank, since banks can predict the aggregate pattern of consumption and thus the pattern of withdrawals over time. Thus, as according to the canonical model of banking in Diamond and Dybvig (1983), liquidity creation is a socially beneficial activity. However, it also makes banks vulnerable to crises. In fact, banks rely on depositors not withdrawing opportunistically. When too many withdrawals take place, then the banking system runs out of resources to service them and a crisis ensues.

When banks create liquidity, they are vulnerable to two types of crises: panic-driven and fundamental. The former are independent of the economy's fundamentals and can be ruled out with appropriate measures, such as a lender of last resort. I assume that these measures are in place and, therefore, focus on the latter type of financial crises. These cannot be avoided costlessly by instituting a lender of last resort, because they are the direct consequence of the economy's fundamentals. In other words, they happen when the banking system becomes insolvent as a consequence of a real shock (Allen and Gale, 1998).

The vulnerability of banks to crises provides a rationale for why banks do not set negative deposit rates. A negative deposit rate gives consumers an incentive to withdraw their deposits and store them in cash. Since excessive withdrawals lead to costly crises, banks choose not to set a negative deposit rate, even if this leads to a compression in their net interest margin.

Because of its focus on the economic impact of changes in the rate of interest, the

model's dynamics are crucial. Hence, this paper generalises the modelling of time in the Diamond-Dybvig model. The economy features an infinite horizon and agents make choices recursively. This generalisation allows us to study shocks that have persistent effects. Moreover, it improves the mapping of the model's variables to the real economy, since in the recursive version of the model endogenous variables such as the deposit rate are a function of the economy's fundamentals and not of time itself.

A first result of the paper is that the economy features two stable balanced growth paths: a good one and a bad one. Along the good BGP, the banking system is healthy and ensures an efficient allocation of consumption. On the other hand, the bad BGP features a failed banking system, unable to provide liquidity. The equilibrium is unique and whether the economy converges to the good or the bad BGP depends on its initial fundamentals.

The main result of the paper is that we can characterise a critical interest-rate level. If the interest rate falls below it, an economy on the good BGP experiences a financial crisis and quickly converges to the bad BGP. This is the tipping point, where low rates generate financial instability. The tipping point is not necessarily zero or negative. Its value depends on the persistence of interest-rate shocks. For permanent reductions in the interest rate, it is strictly positive, because banks need to earn a strictly positive net interest margin on average in order to be solvent. The less persistent the change in the interest rate, the lower the tipping point. In other words, if expectations are for the interest rate to go up in the near future, then the banking system can survive a lower interest rate today. In a numerical exercise, I find that banks can withstand a -25% annualised interest rate for one quarter, if the interest rate is expected to go back to the 2% BGP level immediately thereafter. On the other hand, if expected to last indefinitely, an interest rate as high as 1% tips the economy into a financial crisis.

The mechanism whereby a persistently low interest rate triggers a financial crisis is interesting in its own right. In fact, the model's recursivity throws into sharper relief a connection between the literature on liquidity creation and a more recent literature that stresses the importance for banks of the franchise value of deposits (Drechsler et al., 2017). A bank that provides liquidity has more deposits outstanding than resources at hand. This is why it is susceptible to crises. A naïve reading of this fact is that the bank has negative equity. However, I show that this is not the case. Appropriately accounting for bank equity in this framework requires recognising that banks have an additional asset. They expect to earn positive net interest margins on their deposits until they are withdrawn in the future. In other words, there is a franchise value of bank deposits. To a first approximation, the franchise value of a unit of deposits is the product of the

expected time to withdrawal of the deposit times the average net interest margin.

A temporary reduction in the interest rate reduces the net interest margin today, once the deposit rate hits zero. But the franchise value of deposits is today's net interest margin plus the present discounted value of future net interest margins. As long as banks are expected to recover today's loss of profitability with higher margins in the future once interest rates are back to normal, the franchise value of deposits does not suffer and, as a consequence, neither does bank equity. The damage to bank equity only takes place when the interest rate is expected to stay low for long enough. When net interest margins are compressed sufficiently far into the future, future net interest margins cannot make up for today's loss of profitability. As a consequence, bank equity falls. And a sufficient fall in equity indicates that the bank is insolvent and sets the stage for a run by depositors.

The paper's results are positive in nature. The interest rate is exogenous and we study how shocks to it affect the economy. Nonetheless, there are two implications for monetary policy that can be taken from the paper. First, an excessively low interest rate tips the economy into a banking crisis. Setting the interest rate below such critical level is costly as it curtails liquidity creation and reduces the productivity of investment. Second, the tipping point below which the model predicts a financial crisis is a function of the outlook for interest rates. The expectation that a reduction in the interest rate is persistent makes it more costly in terms of financial stability. Thus, the model highlights a tension between different dimensions of monetary policy: interest-rate setting and forward guidance. Forward guidance is the commitment by a central bank to keep the interest rate at a low level for an extended period of time. It may constrain how low the central bank can cut the interest rate today.

Related literature. Other papers explored the effect of low rates in an economy where banks are subject to a zero lower bound on the deposit rate. These papers take the view of banks as credit intermediaries ([Gertler and Kiyotaki, 2010](#)). Hence, they find that the loss of bank profitability due to the low-rate environment reduces lending. In particular, there is a reversal rate below which an interest-rate cut becomes contractionary. [Eggertsson et al. \(2019\)](#) finds that the reversal rate is zero. In a model with a more detailed banking sector, [Brunnermeier and Koby \(2018\)](#) finds that the reversal rate can be either positive or negative, depending on a series of characteristics of bank balance sheets. [Boissay et al. \(2016\)](#) also studies the impact of low rates on bank lending. However, the key friction is not the zero lower bound on the deposit rate but agency problems in the interbank market, which are exacerbated by the low interest rate.

Clearly, banks play multiple roles in the economy. There has been an attempt to quantify what activity drives bank value most. [Egan et al. \(2017\)](#) finds that, while both productivity in deposit-taking and productivity in screening and monitoring explain large shares of variation in bank value, the former explains more of it.

The canonical model of liquidity creation by banks is laid out in [Diamond and Dybvig \(1983\)](#). A more modern description of the model can be found in [Farhi et al. \(2009\)](#). A strand of the literature on liquidity creation focused on fundamental-driven financial instability ([Calomiris and Gorton, 1991](#); [Calomiris and Mason, 2003](#); [Allen and Gale, 1998, 2000](#)). In line with it, my paper analyses financial instability resulting from real shocks, with a focus on interest-rate shocks. Interestingly, [Allen and Gale \(1998\)](#) makes the point that bank liquidation as a consequence of bad fundamentals can be optimal. This is true under the assumption that liquidation is costless. [Segura and Suarez \(2017\)](#) is the first attempt to capture liquidity creation in a recursive setting.

Alternative views of liquidity have been put forth in different literatures. The very influential literatures started by [Kiyotaki and Moore \(1997\)](#) and [Holmström and Tirole \(1997\)](#) focus on the inefficiently low quantity of liquidity produced in environments with limited commitment.

A recent paper that studies the macroeconomic effects of financial crises and is therefore closely related to the present one is [Gertler et al. \(2019\)](#). Their model successfully matches the large and adverse macroeconomic impact of a financial crisis. Unlike my paper, banks do not perform liquidity creation and financial instability is caused by panics rather than economic fundamentals.

My paper is related to the very large literature on the lower bound on the policy rate. The founding idea of this literature is that the presence of currency makes it infeasible for the short-term risk-free interest rate to go below zero ([Krugman, 1998](#); [Eggertsson and Krugman, 2012](#)). Since the global financial crisis, policy rates in many advanced economies have fallen to large negative values. While it is plausible that, due to banks' ability to store currency, a lower bound on the policy rate exists, the current evidence is that the zero lower bound on the deposit rate is a friction that comes into play before the policy rate reaches said lower bound.

The empirical literature evaluated side effects of low interest rates on banks. [Borio et al. \(2017\)](#) finds that lower policy rates and flatter yield curves predict lower net interest income. [Altavilla et al. \(2018\)](#) finds that a lower policy rate compresses net interest margins. However, bank profits in the short run increase due to more lending and improved counterparty risk. Empirical research in the side effects of negative interest rate policies found that they reduce the equity value of banks ([Ampudia and](#)

Van den Heuvel, 2018), they make banks take on more risk (Basten and Mariathan, 2018; Heider et al., 2018), and they transmit less well to lending rates (Eggertsson et al., 2019) but not by much (Amzallag et al., 2018).

Finally, this paper is related to the mostly empirical literature that studies the financial characteristics of bank deposits. This literature finds that deposits are akin to long-term fixed-rate debt (Di Tella and Kurlat, 2017; Drechsler et al., 2018). Thus, banks' franchise value of deposits is subject to interest-rate risk.

Paper outline. In the next section, I describe the environment and characterise the efficient allocation. Next, I set up the optimisation problems of the decentralised economy. Section 4 solves for the decentralised equilibrium. Section 5 characterises the balanced growth paths of the economy. The following section shows that bank value is determined by the franchise value of deposits. In section 7, I show that the economy can move from the good balanced growth path to the bad one in response to a large enough interest-rate shock. I call this event a banking crisis and characterise the critical interest-rate level below which a banking crisis occurs. The last section contains the numerical exercise. Proofs of propositions, lemmas and corollaries are in the appendix.

2 Preferences and Technology

The economy is inhabited by a unit mass of infinitely-lived consumers who enjoy consumption according to utility function

$$\mathcal{U}(\{C_t\}_{t=s}^{+\infty}) = \lim_{T \rightarrow +\infty} \sum_{t=s}^T \theta_t \cdot u(C_t) + \left(1 - \sum_{t=s}^T \theta_t\right) \cdot u(C_{T+1}). \quad (1)$$

The felicity function u features constant relative risk aversion $1/\alpha > 1$. The random variable θ_t represents an idiosyncratic liquidity shock, which takes on values 0 or 1, and evolves according to the following process

$$Pr(\theta_{t+1} = 1 | \{\theta_j\}_{j=s}^t) = \begin{cases} \phi & \text{if } \sum_{j=s}^t \theta_j = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The liquidity shock can only hit a consumer once. If the consumer has not been hit yet, then in any given period there is a constant probability ϕ that she will be hit. A consumer that is hit by the liquidity shock before another consumer is said to be an earlier type. The other consumer is a later type. An individual's idiosyncratic liquidity

shock is privately observed.

For $T = s$, the utility function nests the standard utility function used in the literature on financial fragility, as in [Diamond and Dybvig \(1983\)](#). In this sense, we can think of the utility function of this paper as an extension of the standard utility function to an infinite horizon.

There are two investment technologies: a productive technology and a storage technology. Both have a one-period maturity. One unit of output invested in the productive technology at time t yields $1 + \rho_t > 1$ units of output at time $t + 1$. The return on the productive technology ρ_t converges to a long-run level $\bar{\rho} > 0$ according to auto-regressive process

$$1 + \rho_t = (1 + \rho_{t-1})^\nu \cdot (1 + \bar{\rho})^{1-\nu}. \quad (3)$$

The parameter $\nu \in (0, 1)$ regulates the persistence of the interest rate or, in other words, the speed with which the interest rate converges to its long-run level. The other investment technology is storage. A unit of output stored today gives a unit of output tomorrow. Notice that the investment technology is simply superior. Hence, in an efficient allocation, no goods would be stored.

Efficiency. Having described the preferences of consumers and the technologies available, we can determine the efficient allocation of consumption. The efficient allocation of consumption to consumers as a function of their realization of the liquidity shock, $\{C_t(\theta)\}_{j=s}^{+\infty}$, maximises the sum of consumers' expected utility

$$\sum_{t=s}^{+\infty} \phi \cdot (1 - \phi)^{t-s} \cdot u[C_t(\theta_t = 1)] \quad (4)$$

subject to the resource constraint

$$Y_t = \phi \cdot (1 - \phi)^{t-s} \cdot C_t(\theta_t = 1) + [1 - \phi \cdot (1 - \phi)^{t-s}] \cdot C_t(\theta_t = 0) + \frac{Y_{t+1}}{1 + \rho_t} \quad \text{for all } t \geq s, \quad (5)$$

where Y_t is the total quantity of resources available for either consumption or investment at time t , and ρ_t is the return of the productive technology, which follows auto-regressive process (3). The social planner is subject to non-negativity constraints

$$C_t(\theta) \geq 0 \quad t \geq s, \quad (6)$$

$$Y_{t+1} \geq 0 \quad t \geq s, \quad (7)$$

and the initial level of resources Y_s is given.

In the efficient allocation, the social planner smooths consumption across states for consumers. However, it is costly to provide this liquidity-risk insurance, because giving consumption to earlier-type consumers implies forgoing future returns on investment. Hence, the optimal path of consumption features partial liquidity-risk insurance and is given by

$$\frac{C_{t+1}(\theta_{t+1} = 1)}{C_t(\theta_t = 1)} = \left[(1 + \rho_s)^{v^{t-s}} \cdot (1 + \bar{\rho})^{1-v^{t-s}} \right]^\alpha \quad t \geq s, \quad (8)$$

$$C_s(\theta_s = 1) = \frac{Y_s}{\phi} \cdot \left[\sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \rho_s)^{\frac{1-v^{t-s}}{1-v}} \cdot (1 + \bar{\rho})^{\frac{1-v^{t-s}}{1-v}}} \right]^{-1}, \quad (9)$$

where $\alpha \in (0, 1)$ is the inverse of the coefficient of relative risk aversion. Since it is inefficient to give consumption to consumers that do not value it, we have

$$C_t(\theta_t = 0) = 0. \quad (10)$$

Notice that full liquidity-risk insurance is a parametric case. When the coefficient of relative risk aversion $1/\alpha \rightarrow +\infty$, then the social planner only cares about providing liquidity-risk insurance, regardless of the cost of doing so.

We did not explicitly take into account that consumers have private knowledge of their type and may choose not to reveal it truthfully to the social planner. In this case, this shortcut comes without loss of generality. Consumers do not have an incentive to mimic other types. In fact, pretending to be an earlier type is sub-optimal, since later types get a weakly higher level of consumption by equation (8). Pretending to be a later type is sub-optimal because consumers derive no utility from consuming after the time of their liquidity shock.

Equations (8), (9) and (10) represent the benchmark for efficiency against which we compare the economic outcomes of the decentralised economy.

3 Decentralised Economy

Two agents inhabit the economy: consumers and banks. Under the assumptions for preferences and technology specified in the previous section, the problems of the two agents are recursive. Hence, we adopt recursive notation.

3.1 Consumers

There is a unit measure of consumers. It is useful to define N as the number of consumers who have not been hit by the liquidity shock yet and therefore have the potential to be hit in the current period. Throughout the paper, I refer to these consumers as active. Those who have been already hit by the liquidity shock are inactive, in that they know they will never again enjoy consumption. It will be useful to express some variables in per-active-consumer terms. Notice that the law of motion for the number of active consumers is given by

$$N' = (1 - \phi) \cdot N, \quad (11)$$

since in every period liquidity shocks hit a share ϕ of the active population.

Consumers are restricted from investing directly in the productive technology. They can either hold their wealth in their deposit account or store it. Holding deposits or storing is the key decision that consumers make. We assume for simplicity that each consumer has an exclusive relationship with one bank, for instance because it is costly to open a deposit account with a new bank.

In what follows, I denote with a hat the variables that are associated with a specific consumer and without hats the aggregate variables.

Value functions. The state variables for an active consumer are the realisation of their idiosyncratic liquidity shock $\hat{\theta}$, which determines whether they have a desire to consume, their individual holdings of deposits \hat{D} and of stored goods \hat{S} . In addition, there is a vector of state variables η that are the same for each individual, such as the interest rate on deposits d . A consumer's value function is given by

$$V_C(\hat{\theta}, \hat{D}, \hat{S}, \eta) = \max_{(\hat{C}, \hat{W}, \hat{W}, \hat{S}', \hat{D}')} \hat{\theta} \cdot u(\hat{C}) + (1 - \hat{\theta}) \cdot \mathbb{E}[V_C(\hat{\theta}', \hat{D}', \hat{S}', \eta')], \quad (12)$$

subject to the following constraints:

$$\hat{S}' + \hat{C} = \hat{S} + \hat{W}, \quad (13)$$

where \hat{W} are actual withdrawals from a consumer's bank deposits. Actual withdrawals may differ from withdrawal demands \hat{W} . In fact, if total demanded withdrawals \tilde{W} exceed the total resources held by the bank B , then the bank's resources are distributed

on a pro-rata basis among the consumers that withdraw, according to

$$\hat{W} = \frac{\min\{\tilde{W}, B\}}{\tilde{W}} \cdot \hat{W}. \quad (14)$$

We can define total actual withdrawals as $W = \min\{\tilde{W}, B\}$. Notice that W/\tilde{W} is the share of demanded withdrawals that banks pay out. If banks have enough resources, it has value 1. If banks' resources are not enough, it is less than 1. A consumer cannot demand to withdraw more than her deposit balance

$$\hat{W} \leq \hat{D}. \quad (15)$$

Withdrawals are flows of deposits. The law of motion for deposits is therefore determined by withdrawals

$$\hat{D}' = (1 + d) \cdot (\hat{D} - \hat{W}). \quad (16)$$

Moreover, consumers are constrained by non-negativity conditions on consumption and storage

$$(\hat{C}, \hat{S}') \geq 0. \quad (17)$$

Optimal withdrawal decision. The key outcome of the consumer's problem is the withdrawal decision, described by policy function

$$\hat{W} = \hat{W}(\hat{\theta}, \hat{D}, \hat{S}, \eta). \quad (18)$$

Notice that \hat{W} can be negative if the consumer deposits the goods that she held in storage. The optimal withdrawing decision is internalised by the bank when deciding the deposit rate to offer. Thus, it plays the role of the incentive-compatibility constraint in the standard Diamond-Dybvig model.

If she is hit by the liquidity shock with $\hat{\theta} = 1$, then the consumer immediately consumes as much as possible, so that

$$\hat{W}(1, \hat{D}, \hat{S}, \eta) = \hat{D}. \quad (19)$$

If the consumer is not hit by the liquidity shock, then she does not consume. She receives no utility from consumption in the current period. She has to decide whether to save her wealth in bank deposits or in storage. We assume that, when absolutely indifferent, the consumer holds deposits rather than stored goods.

Lemma 1. *The value of holding deposits for an active consumer with $\theta = 0$ is λ_D , given by*

$$\frac{\lambda_D}{1+d} = \mathbb{E} \left[\phi \cdot \frac{W'}{\tilde{W}'} \cdot u'(\hat{C}') + (1-\phi) \cdot \max \left\{ \lambda'_D, \frac{W'}{\tilde{W}'} \lambda'_S \right\} \right], \quad (20)$$

The value of storing for an active consumer with $\theta = 0$ is λ_S , given by

$$\lambda_S = \mathbb{E} \left[\phi \cdot u'(\hat{C}') + (1-\phi) \cdot \max \left\{ \lambda'_D, \lambda'_S \right\} \right]. \quad (21)$$

A late consumer's withdrawing behaviour is given by

$$\hat{W}(0, \hat{D}, \hat{S}, \eta) = \begin{cases} -\hat{S} & \text{if } \lambda_D \geq \lambda_S, \\ \hat{D} & \text{if } \lambda_D < \lambda_S. \end{cases} \quad (22)$$

Lemma 1 points out the criteria according to which an active consumer who is not hit by the liquidity shock decides whether to hold deposits or store. In what follows, we use it to work out the consumers' equilibrium behaviour. The following corollaries give sufficient conditions under which active consumers withdraw all of their wealth from the banking system.

Corollary 1. *At a given date s , if $d_s < 0$, then we have that $\hat{W}(0, \hat{D}_s, \hat{S}_s, \eta_s) = \hat{D}_s$*

Regardless of everything else, it is optimal for a consumer who is offered a negative deposit rate to withdraw from the bank and store her wealth, since the one-period return on storage is higher. Of course, this will be a good reason for the bank not to set a negative deposit rate, in the first place.

Corollary 2. *At a given date s , if $\mathbb{E}_s(D_t) > 0$ and $\mathbb{E}_s(B_t) = 0$ for all $t \geq s+1$, then we have $\hat{W}(0, \hat{D}_s, \hat{S}_s, \eta_s) = \hat{D}_s$.*

Corollary 2 shows that if the consumer expects the bank not to have any resources for the whole future but to have outstanding deposits, then it is optimal for the individual to pull out her deposits.

Expectations. At this point, I must make an assumption on consumers' expectations, in order to rule out runs determined solely by pessimism. In other words, I am selecting the equilibrium with fundamental runs. In this model, the expectation of a bank run can be self-fulfilling, regardless of fundamentals. However, in this paper I am interested in analysing fundamental runs. These are unavoidable runs. That is, they occur even if depositors are as optimistic as possible given the bank's fundamentals. I

formalise this notion by endowing consumers with expectations that, while rational, are the most favourable to the banking system.

At any point in time s , the consumer's expectations $\{E_s(B_t), E_s(C_t), E_s(W_t)\}_{t=s}^{+\infty}$ are rational, in the sense that

$$E_s(B_t) = B_t, \quad (23)$$

$$E_s(C_t) = C_t, \quad (24)$$

$$E_s(W_t) = W_t \quad (25)$$

is true. Subject to these constraints, expectations maximise the value of deposits over storage

$$\frac{\lambda_{D,t}}{\lambda_{S,t}}. \quad (26)$$

Notice that in a model in which under rational expectations the equilibrium is determined uniquely, then this maximisation is meaningless, since the choice set would be a single point. An interpretation of this restriction on expectations is that deposit insurance or a lender of last resort eliminate the purely panic-based runs on banks by coordinating expectations in the best possible way. The remaining runs are not due to coordination failures but to bad fundamentals.

Distribution in the state space. The function $f(\hat{D}, \hat{S})$ defines the distribution of active consumers in the state space. Henceforth, I assume that there is no heterogeneity among consumers ex ante. This means that the distribution of consumer deposits and stored goods is degenerate with $f(\hat{D}, \hat{S}) = 0$ for all $(\hat{D}, \hat{S}) \neq (D, S)$. In this model, if the state of the economy in a given period features no heterogeneity across consumers, then there will be also no heterogeneity across the consumers who are active in the following period. In fact, identical consumers have symmetric withdrawal behaviour. Moreover, they receive the same deposit rate if they do not withdraw, and they receive the same amount if they withdraw. Substituting pro-rata distribution of bank resources in case of bank failure with a sequential-service constraint would make it necessary to keep track of heterogeneity among consumers as an additional state variable, because in case the bank fails to repay all demanded withdrawals some active agents get full repayment of their deposits and some receive nothing.

3.2 Banks

Banks finance themselves offering demandable deposit contracts. This contract has two main characteristics: (1) deposits are non-contingent and (2) deposits are

convertible on demand into goods. These characteristics are in conflict, in case the bank does not have enough resources to service all the withdrawals that are demanded. In this case, a bank-failure protocol is followed. The bank distributes all its resources to withdrawing depositors on a pro-rata basis. The rest of the deposits remains. The bank must pay them in the future with any assets it may come to have.

The deposit contract specifies a deposit rate d for every possible state of the world. In particular, I assume that the bank sets the deposit rate to maximise the sum of active depositors' expected utility

$$\phi \cdot V_C(1, D, S, \eta) + (1 - \phi) \cdot V_C(0, D, S, \eta). \quad (27)$$

This is the standard assumption in the literature on bank runs. The justification for this assumption is that it is the most favourable deposit contract from the consumer's perspective. Hence, if profit-maximising banks competed for deposits at the beginning of time, this is the contract that would prevail.

The bank has per-active-consumer resources B at its disposal, with which it pays out withdrawals W and invests in the productive technology. The productive technology has a real return ρ .

$$(1 - \phi) \cdot \frac{B'}{1 + \rho} + W = B. \quad (28)$$

The total quantity of actual withdrawals is the smallest value between the total amount of demanded withdrawals and the bank's resources,

$$W = \min\{\tilde{W}, B\}. \quad (29)$$

If the bank cannot pay all the demanded withdrawals, the bank cannot choose its deposit rate.

$$d \cdot (\tilde{W} - W) = 0. \quad (30)$$

If the bank failed to honour its debt, the deposit rate is fixed to zero. This assumption is at the heart of the bank-failure protocol. It prevents banks from failing to pay their deposits and then wiping them out by setting a negative deposit rate. It is important to notice that, if the bank has not failed, then there is in principle no lower bound on the deposit rate. As we will see, the zero lower bound on the deposit rate arises endogenously from consumers' threat to move their wealth out of the banking system into storage.

We make use of the characterisation of withdrawal behaviour from the previous

subsection to write total demanded withdrawals at any given point in time:

$$\tilde{W} = \phi \cdot D + (1 - \phi) \cdot \hat{W}(0, \hat{D}, \hat{S}, \eta). \quad (31)$$

The bank takes the effect of its actions on consumer withdrawing behaviour into account, when optimising. In particular, the behaviour of the consumers not hit by the liquidity shock, who withdraw according to $\hat{W}(0, \hat{D}, \hat{S}, \eta)$, is key. This behaviour is analysed in detail in lemma 1. For instance, we know that a negative deposit rate leads to withdrawal of all bank deposits, including from patient consumers, who are better off storing. As a consequence, the bank avoids setting a negative deposit rate. In the Diamond-Dybvig model the incentive-compatibility constraint plays the same role constraining the bank.

Individual stored goods and deposits evolve over time according to their respective laws of motion (13) and (16).

Lemma 2. *Suppose consumers are identical ex-ante, in the sense that their distribution in the state space is given by $f(\hat{D}, \hat{S}) = 0$ for $(\hat{D}, \hat{S}) \neq (D, S)$. Then, the bank's value function is given by*

$$V(D, S, B, \rho) = \max_{(d, W, \tilde{W}, D', S', B')} \phi \cdot u\left(\frac{W}{\tilde{W}} \cdot D + S\right) + (1 - \phi) \cdot V(D', S', B', \rho'), \quad (32)$$

subject to constraints:

$$(1 - \phi) \cdot \frac{B'}{1 + \rho} + W = B, \quad (28)$$

$$W = \min\{\tilde{W}, B\}, \quad (29)$$

$$\tilde{W} = \phi \cdot D + (1 - \phi) \cdot \hat{W}(0, D, S, \eta), \quad (31)$$

$$D' = (1 + d) \cdot \left[D - \frac{W}{\tilde{W}} \cdot \hat{W}(0, D, S, \eta) \right], \quad (33)$$

$$S' = S + \frac{W}{\tilde{W}} \cdot \hat{W}(0, D, S, \eta), \quad (34)$$

constraint on deposit rates (30) and the exogenous process for interest rates (3).

Notice that W/\tilde{W} is the share of demanded withdrawals that are actually paid out. From equation (29), we can see that it is smaller than 1 if the bank failed. Otherwise, it is equal to 1.

4 Dynamics of the Model

The deposit contract determines the deposit rate as a function of the state of the economy and of the model's parameters. In turn, the deposit-rate decision determines how the economy evolves and whether it is subject to financial crises. In this section, we study it and the allocation that it implies.

Deposit-rate determination. The bank executes over time its deposit contract. Accordingly, the deposit rate is set to optimise the problem detailed in lemma 2.

Lemma 3. *Given state variables (D, B, S, ρ) , the equilibrium deposit rate is given by*

$$1 + d = \max \left\{ (1 + \rho)^\alpha \cdot \frac{\left(\frac{B+S}{D+S} - \phi \right)}{\phi \cdot \sum_{t=1}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{(1-\alpha) \cdot \frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{(1-\alpha) \left(t - \frac{1-\nu^t}{1-\nu} \right)}}}, 1 \right\}. \quad (35)$$

Equation (35) specifies the deposit rate that the bank offers at a given point in time as a function of the state of the economy. The most noteworthy feature of the deposit-rate decision is a kink at zero. In principle, for some states of the world, for example if the interest rate is low, the bank would like to set a negative deposit rate. However, it refrains from doing so, because consumers threaten to move their wealth into storage. I will refer to this discontinuity as the zero lower bound on the deposit rate. Notice that the model generates it endogenously, because banks fear failure due to excessive withdrawals. In the region where the deposit rate is away from zero, the bank sets it at a higher level when the current interest rate ρ and the long-run interest rate $\bar{\rho}$ are higher. Moreover, it sets it higher, when the bank has many resources relative to outstanding deposits. We can visualise the deposit rate in figure 1, for the special case in which there are no goods in storage (i.e., $S = 0$) and the interest rate is at its long-run level $\rho = \bar{\rho}$.

Equilibrium withdrawal behaviour. Depositors react to the state of the economy and the deposit rate on offer by deciding how much to withdraw from the banking system. Active consumers who are hit by the liquidity shock have full withdrawal as dominant strategy. The decision of active consumers who are not hit by the liquidity shock is interesting. They weigh the risk of keeping their wealth in the banking system against the higher return ensured by deposits over storage and decide how much to withdraw as explained in the following lemma.

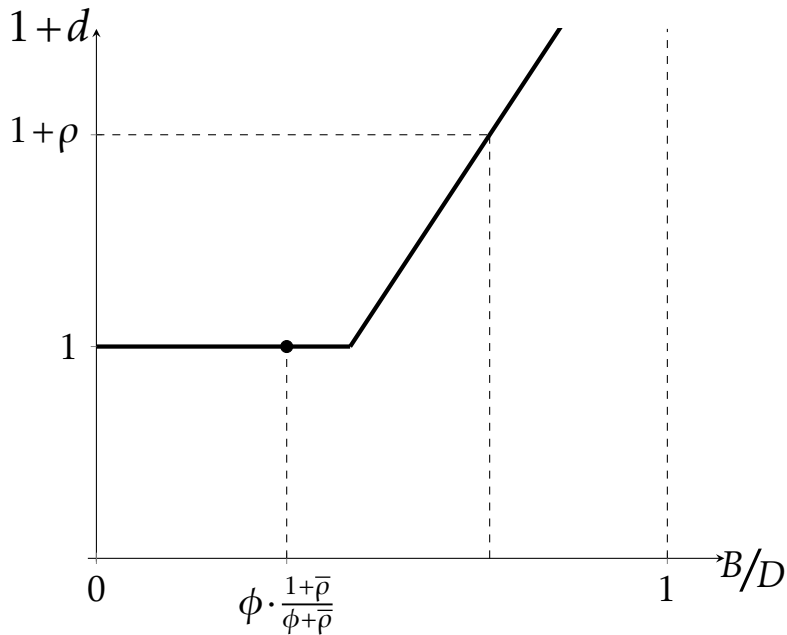
Lemma 4. *Given state variables (D, B, S, ρ) , the equilibrium withdrawal behaviour is given by*

$$\hat{W}(0, D, S, \eta) = \begin{cases} -S & \text{if } \frac{B+S}{D+S} \geq \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}, \\ D & \text{otherwise.} \end{cases} \quad (36)$$

Patient depositors withdraw according to a threshold strategy. If the bank were expected to honour its promises under any circumstance, then no consumer should use the storage technology. However, if the bank has fewer resources per unit of deposit than a threshold, then the consumer can be sure that at some point in the future the bank will not be able to service all withdrawals, even if depositors wait for their liquidity shock to withdraw. If a patient depositor expects that the bank will fail in the next time period, then it is in her best interest to anticipate the crisis and withdraw her deposits today. In equilibrium, all depositors withdraw all their deposits immediately. If the bank is above the threshold, the bank offers a (weakly) positive deposit rate up until the infinite future. Thus, it is in the consumers' interest to keep their deposits in the banking system. Given that purely pessimism-driven crises are ruled out, there are no financial crises. In figure 1, the crisis threshold is represented on the x axis. To the left of it, the bank fails.

State-variable dynamics. Given the state of the economy, the bank's deposit-rate decision and consumer's equilibrium withdrawal decision, we can determine the trajectory of the economy's state variables.

Figure 1: Deposit rate with $S = 0$ and $\rho = \bar{\rho}$.



The deposit rate accumulates and increases the stock of deposits. If the bank fails, some deposits are not paid out since the bank exhausts its resources. However, deposits are not wiped out. If in the future the bank ever had an asset because for instance someone deposited it in the bank, then depositors would still have a claim to a share of it.

$$D' = \begin{cases} (1+d) \cdot (D+S) & \text{if } \frac{B+S}{D+S} \geq \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}, \\ \frac{D-B}{1-\phi} & \text{otherwise.} \end{cases} \quad (37)$$

Bank resources remain positive as long as the net outflow of resources, given by withdrawals, is smaller than the resources held by the bank. Bank failure completely exhausts bank resources.

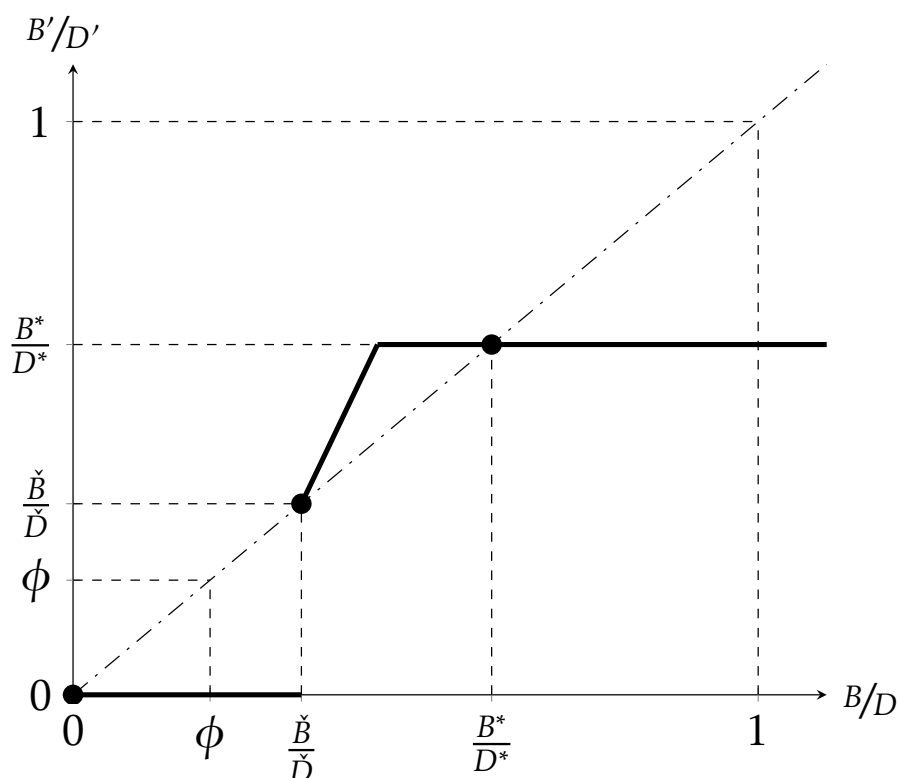
$$B' = \begin{cases} \frac{1+\rho}{1-\phi} \cdot [B - \phi \cdot D + (1-\phi)S] & \text{if } \frac{B+S}{D+S} \geq \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}, \\ 0 & \text{otherwise.} \end{cases} \quad (38)$$

Unless banks fail, the stored goods are deposited in the banking system, because banks offer better returns. On the other hand, bank failure implies that all resources are taken out of the banking system and stored.

$$S' = \begin{cases} 0 & \text{if } \frac{B+S}{D+S} \geq \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}, \\ B+S & \text{otherwise.} \end{cases} \quad (39)$$

The dynamics of B/D are represented in figure 2 for the special case in which $S = 0$ and $\rho = \bar{\rho}$. For very high levels of resources per deposit, the bank sets a high deposit rate in order to boost the consumption of the earlier types who have a higher marginal utility of consumption. This reduces bank resources per unit of deposit in the next period. On the other hand, banks with lower levels of resources relative to deposits reduce the deposit rate on offer. This leads to more resources in the following period relative to deposits. However, there is a lower limit to the deposit rate and, as a consequence, a kink in the law of motion for resources over deposits. In the region of state space in which the deposit rate is zero, the bank fails to accumulate resources relative to deposits as quickly as it would like to. Even further to the left on the figure's x-axis, the bank is unable to pay a non-negative deposit rate at every future date. Hence, consumers run and the bank's resources go down to zero.

Figure 2: Law of motion for $\{B/D\}$ with $S = 0$ and $\rho = \bar{\rho}$.



5 Balanced Growth Paths

In the long run, this economy converges to a balanced growth path. The question is which one. As it turns out, there are multiple balanced growth paths, to which the economy converges depending on its initial state. This section studies the nature of the balanced growth paths and the process of convergence towards them.

We define a BGP in the following way:

Definition 1 (BGP). *A balanced growth path is an equilibrium trajectory of the state variables X_t along which they grow at a constant rate g_X , according to*

$$X_t = (1 + g_X)^{t-s} \cdot X_s \quad \text{for any } t. \quad (40)$$

It turns out that this model has three BGPs. There is a level of resources, relative to deposits, with which banks can implement the first-best efficient allocation. This characterises one of the BGPs. However, since deposits are non-contingent and deposit rates feature a lower bound, it is possible for the bank to remain stuck in a situation with too many deposits outstanding backed by too few resources in the bank. As consumers

become aware of this, they run on the bank and the economy remains at a bad BGP without a working banking system. A knife-edge case between these two BGPs exists too.

Proposition 1 (Characterisation of BGPs). *The economy features three types of balanced growth paths:*

1. The so-called **good BGPs** have growth rates

$$1 + g_D^* = 1 + g_B^* = (1 + \bar{\rho})^\alpha. \quad (41)$$

The initial conditions (B_s^*, D_s^*, S_s^*) are such that

$$\frac{B_s^*}{D_s^*} = \frac{(1 + \bar{\rho})^{1-\alpha}}{(1 + \bar{\rho})^{1-\alpha} - (1 - \phi)} \cdot \phi, \quad (42)$$

$$S_s^* = 0. \quad (43)$$

2. The so-called **unstable BGP** has growth rates

$$\check{g}_B = \check{g}_D = 0. \quad (44)$$

The initial conditions $(\check{B}_s, \check{D}_s, \check{S}_s)$ are such that

$$\frac{\check{B}_s}{\check{D}_s} = \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}, \quad (45)$$

$$\check{S}_s = 0. \quad (46)$$

3. The so-called **bad BGPs** have growth rates

$$1 + g_D^\dagger = \frac{1}{1 - \phi}, \quad (47)$$

$$g_S^\dagger = 0. \quad (48)$$

The initial conditions $(B_s^\dagger, D_s^\dagger, S_s^\dagger)$ are such that

$$\frac{S_s^\dagger}{D_s^\dagger} < \phi \cdot \frac{1 + \bar{\rho}}{(1 - \phi) \cdot \bar{\rho}}, \quad (49)$$

$$B_s^\dagger = 0. \quad (50)$$

Figure 2 helps us to conceptualise the existence of three types of balanced growth paths. The state space can be summarised with the ratio of bank resources to deposits. If this ratio is higher than the value along the good BGP, then the bank optimally sets a higher deposit rate in order to give more consumption to earlier consumers who have relatively high marginal utility of consumption. If the ratio is below, the bank reduces the deposit rate in order to reduce the consumption of earlier types. However, the bank faces a constraint in reducing the deposit rate. There is a lower bound at zero. If the bank has few resources relative to its deposits, it would like to set a negative deposit rate to immediately move up to the good BGP. But it cannot do so. Hence, convergence to the good BGP is slow. There is a point where the convergence to the good BGP takes an infinite amount of time. In other words, the bank can pay a zero deposit rate forever without accumulating or losing resources relative to its outstanding deposits. This is the unstable BGP. If it were to move by an infinitesimal amount to the left of the unstable BGP and set a zero deposit rate forever, it would be certain that in some future period the bank would lack the resources to pay back deposit-holders in full. It follows that consumers are better off running immediately on the bank. And the economy converges immediately to the bad BGP, along which consumers store their goods because the banking sector has failed.

It is interesting to look at the deposit rate that the bank offers along the three BGPs. Along a good BGP, the bank offers a deposit rate equal to

$$1 + d^* = (1 + \bar{\rho})^\alpha. \quad (51)$$

The deposit rate is less than the interest rate that the bank earns on its resources. Hence, the bank's interest-rate margin on deposits is strictly positive. Notice that this expression corresponds to the optimal level of liquidity-risk insurance of the standard Diamond-Dybvig model. With the interest-rate margin, the bank subsidises earlier-type depositors at the expense of later-type depositors. From an ex-ante perspective, this is optimal for consumers.

The unstable BGP is the BGP along which the bank's deposit-rate setting is constrained with $\check{d} = 0$. Yet, the bank has just enough resources to make good on its promises until the infinite future. The deposit rate along the bad BGP is undetermined. Whatever deposit rate the bank offers, consumers run on the bank. Consumers rationally predict that the bank does not have enough resources to pay at least a zero return on its deposits.

Given an initial condition for the economy's resources $B + S = Y$, we can rank the BGPs in terms of welfare. The good BGP replicates the efficient allocation that we

analysed in section 2. Therefore, it is strictly better than the two other BGPs. Both the unstable and bad BGP offer a flat inter-temporal profile of consumption to consumers. But, the unstable BGP offers more resources over time, since resources have better returns if held by banks rather than stored. The inter-temporal profile of consumption associated with the unstable BGP is therefore on a higher level and dominates the bad BGP's in terms of welfare.

As a function of the current state of the economy, we can identify what BGP the economy converges to over time. The following lemma lays out the result.

Lemma 5 (Convergence). *Consider a possible state of the economy (B, D, S, ρ) .*

If

$$\frac{B+S}{D+S} > \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}, \quad (52)$$

then the trajectory converges to the good BGP.

If

$$\frac{B+S}{D+S} = \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}, \quad (53)$$

then the trajectory converges to the unstable BGP.

Otherwise, the trajectory converges to the bad BGP.

6 Franchise Value of Deposits and Bank Equity

The economy that we described so far, in which banks have more deposits outstanding than resources, challenges our notion of balance sheets. Our intuition is that the value of resources and liabilities must be matched, or else the bank's equity is negative. In what follows, I argue that this intuition is valid. Yet, the banks in this model do not necessarily have negative equity. They only have negative equity along the bad BGP.

A missing asset accounts for this puzzle: the bank's deposit franchise. Over the lifetime of a deposit, the bank expects to make a sequence of positive net interest margins. This gives value to the bank and reassures depositors that they will be paid back in full. In other words, the bank is solvent even if it has fewer physical resources than outstanding deposits, as long as the value of its deposit franchise is sufficiently high. If the franchise value of deposits falls, for instance as a consequence of a fall in

the interest rate, then indeed a bank's equity may turn negative and, as I show in this section, a financial crisis ensues.

I start by postulating an accounting identity

$$B + F \cdot D \equiv E + D. \quad (54)$$

We know that B are physical resources held by the bank and D are the deposits it owes. $F \cdot D$ is the value of the deposit franchise. Banks fund themselves with deposits on which, in general and indeed along any good BGP, they pay a below-market interest rate. The capitalised value of these interest differentials is the franchise value. I define the franchise value of deposits for banks at time s as

$$F_s \cdot D_s \equiv \sum_{t=s+1}^{+\infty} \frac{W_t}{D_t} \left[\prod_{j=s}^{t-1} \left(1 - \frac{W_j}{D_j}\right) - \prod_{j=s}^{t-1} \left(1 - \frac{W_j}{D_j}\right) \cdot \frac{1 + d_j}{1 + \rho_j} \right] \cdot D_s. \quad (55)$$

The formula takes into account the fact that deposits are withdrawn from the bank in each period with probability W/D . Notice that along a good BGP, the withdrawal probability of a unit of deposits is given by the constant ϕ . With this in mind, we can interpret the formula as the average interest-rate margin that the bank makes over the lifetime of a unit of deposits, multiplied times the total quantity of deposits. To gain more intuition, we can carry out a first-order approximation of F_s around the good BGP, where $\hat{\rho}$ indicates a log-deviation from the BGP value $\bar{\rho}$ and \hat{d} indicates a log-deviation of the deposit rate from the BGP value d^* . The result is

$$F_s \approx \frac{(1 - \phi) \cdot [(1 + \bar{\rho})^{1-\alpha} - 1]}{(1 + \rho)^{1-\alpha} - (1 - \phi)} + \frac{\phi \cdot (1 + \bar{\rho})^{1-\alpha}}{(1 + \bar{\rho})^{1-\alpha} - 1} \sum_{t=s+1}^{+\infty} \left(\frac{1 - \phi}{(1 + \bar{\rho})^{1-\alpha}} \right)^{t-s} \cdot (\hat{\rho}_{t-1} - \hat{d}_{t-1}). \quad (56)$$

The franchise value along the good BGP is positive and increases with the bank's net interest margin. An even simpler formulation can be obtained for $\bar{\rho} \rightarrow 0$. Then, we have that

$$F_s \approx \sum_{t=s+1}^{+\infty} (1 - \phi)^{t-s} \cdot (\hat{\rho}_{t-1} - \hat{d}_{t-1}). \quad (57)$$

The franchise value is given by the net interest margins in the future adjusted for the expected lifetime of deposits.

E is the residual that makes the accounting identity (54) hold for any value of the other variables. We can think of it as equity. That is the value of the bank for a fictitious residual claimant. It is a meaningful concept to interpret the model, as shown by the

Figure 3: Bank balance sheet along the good BGP.

Assets	Liabilities
B	D
$F \cdot D$	

following proposition.

Lemma 6. *If the economy converges to the bad BGP, then we have that $E < 0$. Otherwise, we have that $E = 0$.*

Since bank failure is synonymous with negative equity, the franchise value of deposits, which largely drives a bank's equity valuation, is key to study financial stability.

It is interesting to notice that, on all trajectories that converge to the unstable or to the good BGP, we have that the bank has exactly zero equity, $E = 0$. This result is tantamount to a zero-profit condition. Subject to the bank being solvent, all the bank's resources are paid out to the depositors. Such contract maximises the depositors' welfare and therefore would emerge in a competitive banking system.

In order to go on and study shocks that lead to financial crises, it is useful at this point to seek necessary and sufficient conditions under which equity is strictly negative. Obviously, a condition that makes use of the accounting identity is $F < 1 - B/D$. However, F is a highly endogenous variable. It turns out that it more helpful to focus on the franchise value of deposits that the bank could achieve by setting the deposit rate to zero from today to the infinite future, given by

$$\bar{F}(\rho) \equiv 1 - \phi - \phi \cdot \sum_{t=1}^{+\infty} \frac{(1 - \phi)^t}{(1 + \rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1 + \bar{\rho})^{t - \frac{1-\nu^t}{1-\nu}}}. \quad (58)$$

In particular, the formula above assumes that the state of the world has $S = 0$. However, this comes without loss of generality for our purposes, since we use the formula to analyse the model starting from the good BGP where no storage takes place. When a bank is creating liquidity, equation (58) describes the greatest possible deposit franchise

for the bank. Of course, the path of deposit rates is determined by the deposit contract. Therefore, in general the bank's value of the deposit franchise is smaller than the maximum. What is important is that it can never be more.

The maximum franchise value of deposits is monotonically increasing in the current interest rate ρ . In fact, the first-order derivative with respect to ρ is given by

$$\frac{\partial \bar{F}(\rho)}{\partial \rho} = \frac{\phi}{(1-\nu) \cdot (1+\rho)} \cdot \sum_{t=1}^{+\infty} \frac{(1-\nu^t) \cdot (1-\phi)^t}{(1+\rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu^t}{1-\nu}}}. \quad (59)$$

The derivative with respect to the persistence of the interest rate is slightly more complicated, given by

$$\frac{\partial \bar{F}(\rho)}{\partial \nu} = -\ln\left(\frac{1+\bar{\rho}}{1+\rho}\right) \cdot \frac{\phi}{(1-\nu)^2} \cdot \sum_{t=1}^{+\infty} \left(\frac{1-\phi}{1+\bar{\rho}}\right)^t \cdot \left(\frac{1+\bar{\rho}}{1+\rho}\right)^{\frac{1-\nu^t}{1-\nu}} \cdot [1-t \cdot \nu^{t-1} - (1-t) \cdot \nu^t]. \quad (60)$$

The sign depends on whether the interest rate is currently below or above the long-run level. If the interest rate is relatively low, then the derivative is negative. An interest rate that remains more persistently below the long-run level implies a lower franchise value of deposits today. In other words, an increase in ν rotates the schedule $\bar{F}(\rho)$ counter-clockwise around the point where $\rho = \bar{\rho}$. We can refer to figure 4 for a graphical representation of schedule $\bar{F}(\rho)$. It represents the case with permanent interest-rate changes. Notice that if the interest rate is permanently at zero, the franchise value of deposits is zero. As the interest rate increases, the franchise value of deposits grows at a decreasing rate.

We can use the definition of $\bar{F}(\rho)$ to find out the interest-rate levels that result in financial crises, as implied by the lemma that follows.

Lemma 7. *Suppose $S = 0$. If and only if*

$$\bar{F}(\rho) < 1 - \frac{B}{D}, \quad (61)$$

then the economy converges to the bad BGP.

The "if" part of the lemma is not particularly interesting. It is obviously a sufficient condition, given that \bar{F} is the maximum level of the franchise value and we know, from lemma 6, that negative equity implies convergence to the bad BGP. The "only if" part is the most interesting. If it is possible for a bank to have non-negative equity by setting the deposit rate to zero forever, then the bank will do so. Banks fail only if it is infeasible

for them to lower deposit rates enough and for long enough.

7 The Tipping Point

We learnt from the previous section that an insufficient value of the deposit franchise leads to a bank run and the economy's convergence to the bad BGP, along which consumers hold their wealth in storage and there is no liquidity-risk insurance. In this section of the paper, we focus on the role that interest-rate shocks may play in bringing about the reduction in the franchise value of deposits that tips the economy into a financial crisis. First, I consider a general state of the economy. This allows us to see the relationship between deposit creation and the economy's financial fragility. Second, I consider the special case of an economy that starts off from the good BGP. In a subsection, I discuss the result and what we can learn from it for monetary policy.

We can characterise a critical value for the interest rate below which a financial crisis ensues. I call this the tipping point.

Proposition 2. *For $S = 0$, there exists a threshold $\underline{\rho}$ such that, if and only if $\rho < \underline{\rho}$, then the economy converges to the bad BGP.*

The threshold is defined by

$$\bar{F}(\underline{\rho}) = 1 - \frac{B}{D}. \quad (62)$$

The tipping point $\underline{\rho}$ is increasing in the extent of liquidity creation. An economy in which banks perform more liquidity creation is more vulnerable to financial crises. An increase in the persistence of interest-rate shocks also makes the economy more vulnerable.

From here on, let's focus on the level of liquidity creation that banks perform along the good BGP and study what is the critical rate of interest below which banks fail.

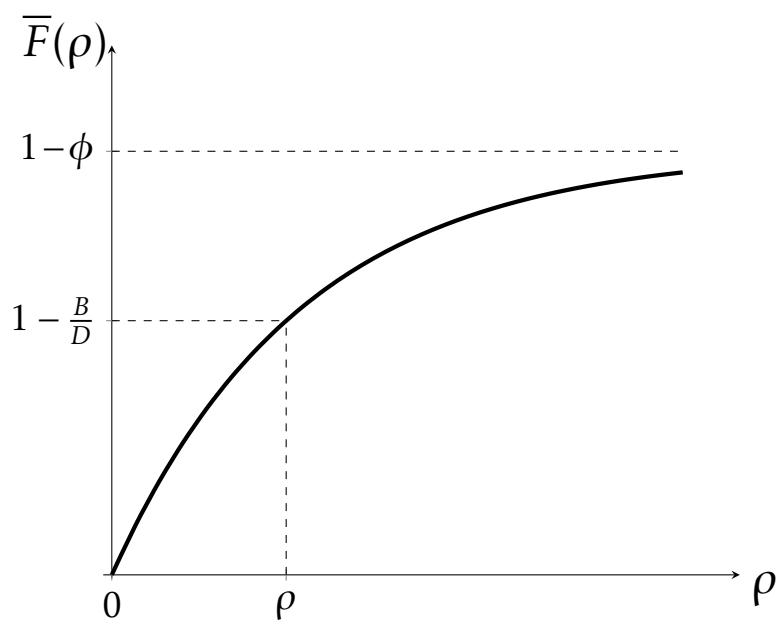
Proposition 3. *Consider an economy on the good BGP. The tipping point $\underline{\rho}^*$ is given by*

$$\bar{F}(\underline{\rho}^*) = (1 - \phi) \cdot \frac{(1 + \bar{\rho})^{1-\alpha} - 1}{(1 + \bar{\rho})^{1-\alpha} - (1 - \phi)}. \quad (63)$$

Corollary 3. *The tipping point is strictly smaller than the long-run interest rate*

$$\underline{\rho}^* < \bar{\rho}. \quad (64)$$

Figure 4: Determination of the tipping point in the case with $\nu = 1$.



A special case of equation (63) is a permanent interest-rate shock (i.e., $\nu \rightarrow 1$). In this case, the expression for the tipping point simplifies to

$$1 + \underline{\rho}^* = (1 + \bar{\rho})^{1-\alpha}. \quad (65)$$

If the interest rate falls permanently below $\underline{\rho}^*$, then the franchise value of deposits goes down. Even if it were to set the deposit rate permanently to zero, future net interest margins would be insufficient to make up for today's compression. The result is a reduction in bank equity and a financial crisis. It is interesting to notice that $\underline{\rho}^* > 0$. Thus, the financial crisis takes place for positive levels of the interest rate. It is especially easy to see the welfare cost of a financial crisis in this case. The bank's productive technology is permanently superior to the storage technology. Yet, consumers run on the banking system and hold their wealth in storage for ever. This is not a sheer panic-driven run. Even under the expectations that are most favourable to the bank, depositors know that, if the bank were to set a zero deposit rate forever, it would eventually exhaust its resources and partly default on its obligations. Since the best course of action for each consumer is to anticipate other depositors in demanding their deposits back, the equilibrium result is an immediate and generalised run on the bank.

The persistence of interest-rate shocks has an unambiguous effect on the tipping point, as summarised below.

Corollary 4. *The tipping point $\underline{\rho}^*$ is strictly increasing in the persistence of the interest rate ν .*

If interest-rate shocks are very persistent, then smaller shocks are enough to bring about a financial crisis. Interestingly, if we consider a central bank that controls the interest rate, then the tipping point is endogenous to interest-rate setting. If agents expect the central bank to keep a low interest rate for long, then the tipping point is higher.

7.1 Tipping point and monetary policy.

The interpretation of the tipping point as a lower limit to interest-rate cuts depends on whether the central bank wishes to avoid the financial crises associated with crossing it. We compare the aggregate utility of consumers in an economy where the interest rate is below $\underline{\rho}$ by a vanishingly small epsilon and thus the economy experiences a financial crisis, with one where the interest rate is precisely equal to $\underline{\rho}$ and therefore no financial crisis takes place.

Proposition 4. *Welfare as a function of the interest rate features a discontinuity at $\underline{\rho}$, as*

$$\lim_{\rho \rightarrow \underline{\rho}^-} V(D, S, B, \rho) < V(D, S, B, \underline{\rho}). \quad (66)$$

Crossing the tipping point and generating a financial crisis has a welfare cost per se, due to the disappearance of liquidity creation and the lower productivity of storage relative to bank investment.

The model does not feature a benefit from interest-rate cuts. Strictly speaking, setting a higher interest rate is always welfare-improving, because it increases the real productivity of investment. Nonetheless, given the discontinuity in welfare at $\underline{\rho}$, a monetary authority would be especially reluctant to cross the tipping point, even if it had a reason to cut the interest rate. In this sense, we can interpret the tipping point as a micro-foundation for an effective lower bound on the interest rate.

A low-for-long policy is the commitment by a central bank to keep the interest rate at a low level for an extended period of time. The same policy is also known as forward guidance. It has been argued that such policy increases monetary stimulus when the effective lower bound on the interest rate is binding and thereby that it is beneficial to the economy (Eggertsson and Woodford, 2003; Bernanke et al., 2019). Indeed, central banks in advanced economies have all deployed forward guidance since 2008. This paper finds that the level of the effective lower bound is in fact endogenous

to expectations of the path of interest rates. Expectations of lower interest rates in the future make an interest-rate cut today costlier. Thus, a tension exists between easing by means of conventional monetary policy and of forward guidance. The more the latter is used, the less space there is for the former.

8 Approximation and Numerical Illustration

It is instructive to approximate equation (63) with a log-linearisation around the good BGP and find that the greatest negative shock that the economy can withstand without giving way to a financial crisis is

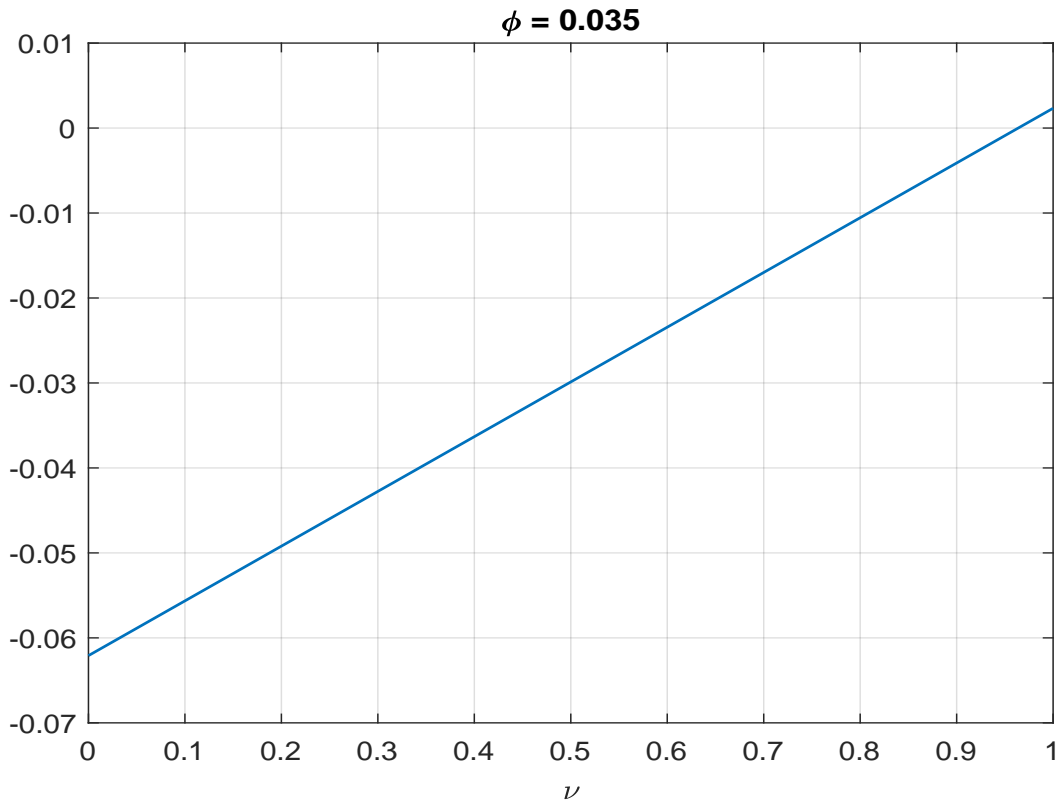
$$\underline{\rho}^* \approx \bar{\rho} - \frac{(1 + \bar{\rho})^{1-\alpha} \cdot [(1 + \bar{\rho})^\alpha - 1]}{(1 + \bar{\rho})^{1-\alpha} - (1 - \phi)} \cdot [1 + \bar{\rho} - (1 - \phi) \cdot \nu]. \quad (67)$$

As long as deposits are expected to last more than zero periods (i.e., $\phi < 1$), the persistence of the interest-rate shock plays a role. The quicker the interest rate is expected to move back to its long-run level (i.e., low ν), the lower the tipping point.

The tipping point depends on the long-term interest rate $\bar{\rho}$ for two reasons with opposite effects. On the one hand, a higher long-term interest rate encourages more deposit creation from the banking system and therefore makes the banking system more vulnerable to interest-rate shocks. On the other hand, after an adverse interest-rate shock took place, expectations that the interest rate will converge to a higher level make the future prospects of the banking system rosier and allow the banking system to withstand a lower interest rate today. The parameter α is the inverse of the coefficient of relative risk aversion. An economy where agents are more risk averse does more liquidity creation. As a consequence, it has a higher effective lower bound and its banks are more vulnerable to negative interest-rate shocks.

I study numerically how low the interest rate can go without a financial crisis taking place, if the economy starts from the good BGP. A time period in this exercise corresponds to a quarter. I set the annualised long-term level of the interest rate to 2%. Setting the coefficient of relative risk aversion equal to 2 (i.e., $\alpha = 0.5$) is standard in the macroeconomic literature. It is tricky to set the parameter ϕ . Notice that $(1-\phi)/\phi$ is the expected lifetime of a deposit. There are some estimates for this in the literature. [Hutchison and Pennacchi \(1996\)](#) and [Musakwa and Schaling \(2019\)](#) both find that the average duration of a dollar deposited at a bank is around 7 years. I set the value of $\phi = 3.5\%$ to match this.

Figure 5: Effective lower bound $\underline{\rho}^*$ as a function of the persistence of the interest rate



If the interest rate shock is purely temporary the banking system can withstand a hugely negative annualised rate of -25% . However, if the interest rate shock is permanent, then the lower the interest rate can go without leading to a financial crisis is 1% . These results show two things. First, the effective lower bound is very dependent on the persistence of the shocks. Second, even in advanced economies, where interest rates have been low in recent years by historical standards, the effective lower bound is far from binding, unless we assume an extremely high persistence of the current low-rate environment.

9 Conclusion

I generalise the Diamond-Dybvig model by extending the time horizon to infinity. This is helpful, because of the paper's aim of studying the effect of changes in the rate of interest, or in other words the opportunity cost of time. Moreover, as a result of

the generalisation, the model becomes nicely recursive and the endogenous variables, which are no longer a function of time but solely of the economy's fundamentals, are easier to interpret.

Two stable balanced growth paths emerge: a good one with functional banks and a bad one with failed banks. BGPs facilitate the interpretation of comparative statics greatly. In fact, studying a shock in the three-date Diamond-Dybvig model has the added complication that the effect depends on the timing of the shock. Does the shock take place at time 0, time 1 or time 2? And what is the interpretation of the different results? Having BGPs solves this problem. As is standard in the macroeconomic literature, we study shocks that hit the economy once it has settled on a balanced growth path.

The paper's main result is that the economy can move from one BGP to the other as a consequence of shocks to the interest rate. In particular, an economy with a healthy banking system can be tipped into the bad BGP by an excessive reduction in the interest rate. We can characterise this tipping point.

The tipping point is not necessarily negative or zero. It depends on the persistence of changes in the interest rate. Banks can withstand a deeply negative interest rate, if the expectation is that it will increase in the near future. On the other hand, a low but positive interest rate expected to last forever may cause the banking system to fail. In a numerical exercise in which I assume the long-term level of the interest rate to be 2% per year, I find that the banking system can withstand a $-25%$ annualised interest rate for a quarter. However, it would fail if the interest were expected to stay at 1% permanently.

The franchise value of deposits, which is the present discounted value of today's and future net interest margins, plays a key role in the balance sheet of banks that create liquidity. If it falls below a given value, then banks are insolvent. This is an aspect of the Diamond-Dybvig model, which is made visible by the recursive setting. The franchise value of deposits is at the heart of the mechanism whereby persistently low rates generate financial instability. Since banks do not offer negative deposit rates out of fear of triggering withdrawals, a large enough reduction in the interest rate today ends up compressing the bank's current net interest margin. However, the effect on the franchise value of deposits depends on expectations of the future path of interest rates. If the interest rate is expected to go back up quickly, agents expect the bank to recover today's lost profitability with wider net interest margins in the future. Hence, the franchise value of deposits is unaffected. On the other hand, if the low-rate environment is expected to persist, it becomes impossible for future net interest margins to be wide enough to compensate for the current squeeze. As a consequence, deposits have a lower

franchise value and bank equity falls. If bank equity turns negative, it means that the bank is insolvent and depositors have an incentive to withdraw all of their deposits immediately.

A direct way of tackling the model's inefficiency is to remove depositors' outside option. If the return on the outside option of consumers were reduced, then the lower bound on the deposit rate would be slackened. This property is common across models that study economies with lower bounds on interest rates. Without the constraint, welfare is higher. It is important to bear this in mind. Cash is the real-world counterpart to this paper's storage technology and proposals to reduce its suitability as a store of value have been put forth (Rogoff, 2017). However, taxing cash holdings is for the time being not a readily available policy option. Thus, it is valuable to explore the effects of changes in the rate of interest, in the context of the current monetary system.

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A Proofs

Proof of lemma 1. Using the fact that $\hat{C}(0, \hat{D}, \hat{S}, \eta) = 0$, we can write the active consumers' problem with $\theta = 0$ as

$$V_C(0, \hat{D}, \hat{S}, \eta) = \max_{(\hat{W}, \hat{D}', \hat{S}')} \phi \cdot u\left(\frac{W}{\hat{W}} \cdot \hat{D}' + \hat{S}'\right) + (1 - \phi) \cdot \mathbb{E}\left[V_C(0, \hat{D}', \hat{S}', \eta')\right] \quad (68)$$

subject to

$$\hat{S} = \hat{S} + \hat{W}, \quad (69)$$

$$\hat{D}' = (1 + d) \cdot (\hat{D} - \hat{W}), \quad (70)$$

$$\hat{S}' \geq 0, \quad (71)$$

$$\hat{W} \geq \frac{W}{\hat{W}} \cdot \hat{D}. \quad (72)$$

The Kuhn-Tucker conditions of this problem confirm the lemma.

Proof of corollary 1. Notice that $w'/\bar{w}' < 1$, since the bank never pays more than the demanded withdrawals. It follows directly that, if $d < 0$, then $\lambda_D < \lambda_S$.

Proof of corollary 2. Notice that $B_t = 0$ and $D_t > 0$ for all $t \geq s + 1$ implies that $w_t/\bar{w}_t = 0$ for all $t \geq s + 1$. A direct consequence is that $\lambda_D = 0$.

Proof of lemma 2. In this proof, I show that the objective function is equivalent to (27), that is the depositors' aggregate welfare at a point in time. Consider a time period s . It is possible to solve forward equation (27) and have

$$\phi \cdot V_C(1, D_s, S_s, \eta_s) + (1 - \phi) \cdot V_C(0, D_s, S_s, \eta) = \sum_{t=s}^{+\infty} \phi \cdot (1 - \phi)^{t-s} \cdot u\left(\frac{W_t}{\bar{W}_t} \cdot D_t + S_t\right). \quad (73)$$

With this in mind, we can write the bank's problem as in the lemma.

Proof of lemma 3. Recall that state variables (B, D, S) evolve over time according to laws of motion (28), (33), (34). Moreover, demanded and actual withdrawals (\tilde{W}, W) are determined in each period by equations (31) and (29).

Part 1. Suppose that at a given date s , the following condition holds:

$$\frac{B_s + S_s}{D_s + S_s} < \phi. \quad (74)$$

From lemma 1, we know that $\hat{W}(0, D_s, S_s, \eta_s) \in [-S_s, D_s]$. Using equations (31) and (29), this implies that $\tilde{W}_s > W_s$. Hence, by equation (30) we can conclude that, under condition (74), $d = 0$.

For what follows, it is useful to notice that $\tilde{W}_s > W_s$ implies that $B_{s+1} = 0$ and $D_{s+1} > 0$. From the laws of motion, we can check that for any withdrawing behaviour

$$\frac{B_{s+1} + S_{s+1}}{D_{s+1} + S_{s+1}} < \phi. \quad (75)$$

Hence, we can iterate this reasoning forward until the infinite future and find that $B_t = 0$ and $D_t > 0$ for all $t \geq s + 1$. Using corollary 2, we have that under condition (74) late types withdraw everything and $\lambda_{D,s} = 0$ and $\lambda_{S,s} > 0$.

Part 2. Suppose that at a given date $s - 1$, the following condition holds:

$$\phi \leq \frac{B_{s-1} + S_{s-1}}{D_{s-1} + S_{s-1}} < \phi \cdot \left[1 + \frac{1 - \phi}{1 + \rho_{s-1}} \right]. \quad (76)$$

First, let's suppose the bank offers $d_{s-1} < 0$. By corollary 1, all late consumers withdraw. Since this implies $\tilde{W}_{s-1} > W_{s-1}$, $d_{s-1} < 0$ is contradicted by the constraint (30). Second, let's consider the bank offering $d_{s-1} \geq 0$. For any deposit rate offered and any withdrawing behaviour, we have that

$$\frac{B_s + S_s}{D_s + S_s} < \phi. \quad (74)$$

This is the case described in part 1 of this proof. Using the results in part 1, we can write that

$$\frac{\lambda_{D,s-1}}{\lambda_{S,s-1}} = (1 + d_{s-1}) \cdot \frac{B_s}{D_s}. \quad (77)$$

Since for any $d_{s-1} \geq 0$ and withdrawing behaviour at time $s - 1$, $(1 + d_{s-1}) \cdot A_s/D_s < 1$, by lemma 1 we have that $\hat{W}(0, D_{s-1}, S_{s-1}, \eta_{s-1}) = D$. Using equations (31) and (29), this implies that $\tilde{W}_{s-1} > W_{s-1}$. Hence, by (30) we have that, under condition (74), $d_{s-1} = 0$.

For what follows, it is useful to notice that $\tilde{W}_{s-1} > W_{s-1}$ implies that $B_s = 0$ and $D_s > 0$. Since at time s condition (74) holds, we can write that $B_t = 0$ and $D_t > 0$ for all $t \geq s$. Using corollary 2, we have that under condition (76) late types withdraw everything and $\lambda_{D,s-1} = 0$ and $\lambda_{S,s-1} > 0$.

Part 3. The argument in part 2 of this proof can be made iteratively for conditions

$$\phi \cdot \sum_{k=0}^{j-1} \frac{(1-\phi)^k}{(1+\rho_{s-j-1})^{\frac{1-\nu k}{1-\nu}} \cdot (1+\bar{\rho})^{k-\frac{1-\nu k}{1-\nu}}} < \frac{B_{s-j} + S_{s-j}}{D_{s-j} + S_{s-j}} < \phi \cdot \sum_{k=0}^j \frac{(1-\phi)^k}{(1+\rho_{s-j})^{\frac{1-\nu k}{1-\nu}} \cdot (1+\bar{\rho})^{k-\frac{1-\nu k}{1-\nu}}} \quad (78)$$

for $j = 2, 3, \dots$

We can summarise the findings of the first three parts of this proof with the following proposition. At a given time s if

$$\frac{B_s + S_s}{D_s + S_s} < \phi \cdot \sum_{t=0}^{+\infty} \frac{(1-\phi)^t}{(1+\rho_s)^{\frac{1-\nu t}{1-\nu}} \cdot (1+\bar{\rho})^{t-\frac{1-\nu t}{1-\nu}}}, \quad (79)$$

then we have that $d_s = 0$. Moreover, $B_t = 0$ and $D_t > 0$ for all $t \geq s+1$. Using corollary 2, we have that under condition (74) late types withdraw everything and $\lambda_{D,s} = 0$ and $\lambda_{S,s} > 0$.

Part 4. At a given time s , let's suppose that condition

$$\frac{B_s + S_s}{D_s + S_s} \geq \frac{\phi}{(1+\rho_s)^\alpha} \cdot \sum_{t=s}^{+\infty} \frac{(1-\phi)^{t-s}}{\left[(1+\rho_s)^{\frac{1-\nu t-s}{1-\nu}} (1+\bar{\rho})^{(t-s-\frac{1-\nu t-s}{1-\nu})} \right]^{1-\alpha}} \quad (80)$$

holds. Notice that in this case $B_s + S_s / D_s + S_s$ is strictly greater than under condition (79). We solve the banks' maximisation problem in lemma 2. Start by guessing that $\hat{W}_t = -S_t$ for all $t \geq s$. We can later verify this. Under this guess, the bank's optimisation problem is to choose $\{d_t, D_{t+1}\}_{t=s}^{+\infty}$, given $\{B_s, D_s, S_s, \rho_s\}$, in order to maximise

$$\phi \cdot u(D_s + S_s) + \cdot \sum_{t=s}^{+\infty} (1-\phi)^{t-s} \cdot u(D_t), \quad (81)$$

subject to the inter-temporal budget constraint

$$\phi \cdot \sum_{t=s+1}^{+\infty} \frac{(1-\phi)^{t-s}}{(1+\rho_s)^{\frac{1-\nu^{t-s}}{1-\nu}} \cdot (1+\bar{\rho})^{t-s-\frac{1-\nu^{t-s}}{1-\nu}}} \cdot D_t = B_s - \phi \cdot D_s + (1-\phi) \cdot S_s, \quad (82)$$

the law of motion of deposits

$$D_t = (1+d_{t-1}) \cdot (D_{t-1} + S_{t-1}) \quad \forall t \geq s+1. \quad (83)$$

The optimality conditions are given by

$$\frac{B_t}{D_t} = \frac{1+\rho_{t-1}}{(1+d_{t-1}) \cdot (1-\phi)} \cdot \left(\frac{B_{t-1}}{D_{t-1}} - \phi \right) \quad \text{for all } t \geq s+1, \quad (84)$$

$$S_t = 0 \quad \text{for all } t \geq s+1, \quad (85)$$

$$1+d_t = (1+\rho_t)^\alpha \cdot \frac{\frac{B_t+S_t}{D_t+S_t} - \phi}{\phi \cdot \sum_{k=1}^{+\infty} \frac{(1-\phi)^k}{\left[(1+\rho_t)^{\frac{1-\nu^k}{1-\nu}} (1+\bar{\rho})^{\left(k-\frac{1-\nu^k}{1-\nu}\right)} \right]^{1-\alpha}}} \quad \text{for all } t \geq s. \quad (86)$$

We must verify the guess on withdrawal behaviour. First, notice that along the path described by the bank's optimality condition $W_t = \tilde{W}_t$ for all $t \geq s$, as long as consumers withdraw according to our guess (remember that expectations are rational and such that the value of deposits is maximised). We can write that this implies the following relation between the value of deposits and the value of storage

$$\lambda_{D,t} = (1+d_t) \cdot \lambda_{S,t} \quad \text{for all } t \geq s. \quad (87)$$

We can see from the optimality condition that, under condition (80), $d_t \geq 0$ for all $t \geq s$. This and the condition found in (87) confirms our guess on withdrawing behaviour, as according to lemma 1.

Part 5. Suppose that at a given time s condition

$$\begin{aligned} \phi \cdot \sum_{t=s}^{+\infty} \frac{(1-\phi)^{t-s}}{(1+\rho_s)^{\frac{1-\nu^{t-s}}{1-\nu}} \cdot (1+\bar{\rho})^{t-s-\frac{1-\nu^{t-s}}{1-\nu}}} &\leq \frac{B_s + S_s}{D_s + S_s} < \\ &< \frac{\phi}{(1+\rho_s)^\alpha} \cdot \sum_{t=s}^{+\infty} \frac{(1-\phi)^{t-s}}{\left[(1+\rho_s)^{\frac{1-\nu^{t-s}}{1-\nu}} (1+\bar{\rho})^{(t-s-\frac{1-\nu^{t-s}}{1-\nu})} \right]^{1-\alpha}} \end{aligned} \quad (88)$$

holds. Consider the optimality conditions found in part 4 of this proof. They imply a negative deposit rate for at least the first period. However, this contradicts the initial guess on withdrawing behaviour, i.e. $\hat{W}(0, D_t, S_t, \eta_t) = -S_t$ for all $t \geq s$. Suppose the bank set $d_s < 0$. Then, by corollary 1 late consumers would withdraw everything and as a result $W_s < \tilde{W}_s$. This is not compatible with $d_s < 0$, by constraint (30). It is therefore optimal for the bank to set $d_s = 0$.

Proof of lemma 4. The proof of this lemma is very closely related to the proof of lemma 3. Parts 1, 2 and 3 of the proof of lemma 3 establish that if at a given time s condition

$$\frac{B_s + S_s}{D_s + S_s} < \phi \cdot \sum_{t=s}^{+\infty} \frac{(1-\phi)^t}{(1+\rho_s)^{\frac{1-\nu^{t-s}}{1-\nu}} (1+\bar{\rho})^{t-s-\frac{1-\nu^{t-s}}{1-\nu}}} \quad (79)$$

holds, then we have that

$$\hat{W}(0, D, S, \eta) = D. \quad (89)$$

Parts 4 and 5 of the proof of lemma 3 establish that if at a given time s 79 does not hold, then we have that

$$\hat{W}(0, D, S, \eta) = -S. \quad (90)$$

Proof of proposition 1. Along a balanced growth path, we have a constant interest rate $\rho = \bar{\rho}$. Remember that the deposit rate is given by (35), and the other state variables evolve according to laws of motion (37), (38) and (39).

At a given time s , let's start by analysing the condition

$$\frac{B_s + S_s}{D_s + S_s} > \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}. \quad (91)$$

Notice that if the condition holds at time s , then it holds for all the following periods, as

$$\frac{B_t + S_t}{D_t + S_t} = \phi \cdot \frac{(1 + \bar{\rho})^{1-\alpha}}{(1 + \bar{\rho})^{1-\alpha} - (1 - \phi)} > \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}} \text{ for all } t \geq s + 1. \quad (92)$$

Using the laws of motion and the equation (92), we can write that

$$1 + g_t^D = 1 + g_t^B = (1 + \bar{\rho})^\alpha \text{ for all } t \geq s + 1. \quad (93)$$

We can have $1 + g_s^D = 1 + g_s^B = (1 + \bar{\rho})^\alpha$ if and only if

$$S_s = 0, \quad (94)$$

$$\frac{B_s}{D_s} = \phi \cdot \frac{(1 + \bar{\rho})^{1-\alpha}}{(1 + \bar{\rho})^{1-\alpha} - (1 - \phi)} > \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}. \quad (95)$$

Second, let us analyse condition

$$\frac{B_s + S_s}{D_s + S_s} = \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}. \quad (96)$$

Notice that if the condition holds at time s , then it holds for all the following periods.

Using the laws of motion and the equation (92), we can write that

$$1 + g_t^D = 1 + g_t^B = 1 \text{ for all } t \geq s + 1. \quad (97)$$

We can have $1 + g_s^D = 1 + g_s^B = 1$ if and only if

$$S_s = 0, \quad (98)$$

$$\frac{B_s}{D_s} = \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}. \quad (99)$$

Third and last, let us analyse condition

$$\frac{B_s + S_s}{D_s + S_s} < \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}. \quad (100)$$

Notice that if the condition holds at time s , then it holds for all the following periods, as

$$\frac{B_t + S_t}{D_t + S_t} = \frac{1 - \phi}{\frac{D_{t-1} + S_{t-1}}{B_{t-1} + S_{t-1}} - \phi} < \frac{B_{t-1} + S_{t-1}}{D_{t-1} + S_{t-1}} \text{ for all } t \geq s + 1. \quad (101)$$

Using the laws of motion, we can write:

$$1 + g_t^D = \frac{1}{1 - \phi} \text{ for all } t \geq s + 1, \quad (102)$$

$$1 + g_t^S = 1 \text{ for all } t \geq s + 1. \quad (103)$$

We can have $1 + g_s^D = 1/(1 - \phi)$ and $1 + g_s^S = 1$ if and only if

$$B_s = 0, \quad (104)$$

$$\frac{S_s}{D_s + S_s} < \phi \cdot \frac{1 + \bar{\rho}}{\phi + \bar{\rho}}. \quad (105)$$

Proof of lemma 5. Remember that in the long run $\rho \rightarrow \bar{\rho}$ due to its law of motion 3. Moreover, the deposit rate is given by (35), and the other state variables evolve according to laws of motion (37), (38) and (39).

At a given time s , let's start considering condition

$$\frac{B_s + S_s}{D_s + S_s} < \phi \cdot \sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \rho)^{\frac{1-\nu}{1-\nu} t-s} \cdot (1 + \bar{\rho})^{t-s - \frac{1-\nu}{1-\nu} t-s}}. \quad (106)$$

Using the laws of motions, we can verify that

$$\lim_{t \rightarrow +\infty} \frac{B_t + S_t}{D_t + S_t} = \lim_{t \rightarrow +\infty} (1 - \phi)^{t-s} \cdot \frac{B_s + S_s}{D_s + S_s} \cdot \frac{1}{1 - \phi \cdot \frac{B_s + S_s}{D_s + S_s} \cdot \sum_{j=s}^t (1 - \phi)^{t-j}} = 0. \quad (107)$$

It follows that the economy converges to a bad BGP.

Now, let's consider condition

$$\frac{B_s + S_s}{D_s + S_s} = \phi \cdot \sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \rho)^{\frac{1-\nu}{1-\nu} t-s} \cdot (1 + \bar{\rho})^{t-s - \frac{1-\nu}{1-\nu} t-s}}. \quad (108)$$

Using the laws of motion, we can find that

$$\frac{B_{s+1} + S_{s+1}}{D_{s+1} + S_{s+1}} = \frac{1 + \rho_s}{1 - \phi} \cdot \left(\frac{B_s + S_s}{D_s + S_s} - \phi \right). \quad (109)$$

Substituting in the condition (108), we have that

$$\frac{B_{s+1} + S_{s+1}}{D_{s+1} + S_{s+1}} = \phi \cdot \sum_{t=s+1}^{+\infty} \frac{(1 - \phi)^{t-s-1}}{(1 + \rho)^{\frac{1-\nu t-s-1}{1-\nu}} \cdot (1 + \bar{\rho})^{t-s-1-\frac{1-\nu t-s-1}{1-\nu}}}, \quad (110)$$

which corresponds exactly to condition (108) one period later. It follows that the economy converges to the unstable BGP.

Third and last, let us consider condition

$$\frac{B_s + S_s}{D_s + S_s} > \phi \cdot \sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \rho)^{\frac{1-\nu t-s}{1-\nu}} \cdot (1 + \bar{\rho})^{t-s-\frac{1-\nu t-s}{1-\nu}}}. \quad (111)$$

Using the laws of motion and substituting in the condition 111, we can find that

$$\frac{B_{s+1} + S_{s+1}}{D_{s+1} + S_{s+1}} > \phi \cdot \sum_{t=s+1}^{+\infty} \frac{(1 - \phi)^{t-s-1}}{(1 + \rho)^{\frac{1-\nu t-s-1}{1-\nu}} \cdot (1 + \bar{\rho})^{t-s-1-\frac{1-\nu t-s-1}{1-\nu}}}. \quad (112)$$

It follows that the economy converges to a good BGP.

Proof of lemma 6. Remember that E is defined by

$$B + F \cdot D \equiv E + D. \quad (54)$$

First, we prove the “if” part of the lemma. By lemma 5, we know that if the economy converges to the bad BGP, then we have that at a given time s

$$\frac{B_s + S_s}{D_s + S_s} < \phi \cdot \sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \rho)^{\frac{1-\nu t-s}{1-\nu}} \cdot (1 + \bar{\rho})^{t-s-\frac{1-\nu t-s}{1-\nu}}}. \quad (113)$$

Lemma 4 shows that under such condition $W_s = D_s$. From the definition of the franchise value of deposits (55), this implies $F_s = 0$. It follows that $E < 0$.

Second, we prove the “only if” part of the lemma. Consider the condition

$$\frac{B_s + S_s}{D_s + S_s} \geq \phi \cdot \sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \rho)^{\frac{1-\nu^{t-s}}{1-\nu}} \cdot (1 + \bar{\rho})^{t-s - \frac{1-\nu^{t-s}}{1-\nu}}}. \quad (114)$$

From lemma 4, we know that in this under such condition $W_s = \phi \cdot D_s - (1 - \phi) \cdot S_s$ and $W_t = \phi \cdot D_t$ for all $t \geq s + 1$. It follows that we can write the franchise value of deposits as

$$F_s = \left(1 + \frac{S_s}{D_s}\right) \cdot \left[1 - \phi - \phi \cdot \sum_{t=s+1}^{+\infty} (1 - \phi)^{t-s} \cdot \prod_{j=s}^{t-1} \frac{1 + d_j}{1 + \rho_j}\right]. \quad (115)$$

Under the initial condition, an inter-temporal budget constraint holds for the bank

$$\phi \cdot \sum_{t=s+1}^{+\infty} \frac{(1 - \phi)^{t-s}}{\prod_{j=s}^{t-1} 1 + \rho_j} \cdot D_t = B_s - \phi \cdot D_s + (1 - \phi) \cdot S_s. \quad (116)$$

We can re-arrange and write it as

$$\phi \cdot \sum_{t=s+1}^{+\infty} (1 - \phi)^{t-s} \cdot \prod_{j=s}^{t-1} \frac{1 + d_j}{1 + \rho_j} = \frac{B_s + S_s}{D_s + S_s} - \phi. \quad (117)$$

Substituting the definition of franchise value and the inter-temporal budget constraint in the definition of equity (54), we get that

$$E_s = 0. \quad (118)$$

Proof of lemma 7. Substituting the definition of $\bar{F}(\rho)$ (58) in equation (61), we have that

$$\frac{B}{D} > \phi \cdot \sum_{t=0}^{+\infty} \frac{(1 - \phi)^t}{(1 + \rho)^{\frac{1-\nu^t}{1-\nu}} \cdot (1 + \bar{\rho})^{t - \frac{1-\nu^t}{1-\nu}}}. \quad (119)$$

Hence, the lemma follows from lemma 5.

Proof of proposition 2 The proposition follows from lemma 7 and the fact that $\bar{F}(\rho)$ is monotonically increasing in ρ .

Proof of proposition 4 At a given time s , if $\rho_s = \underline{\rho}_s$, by definition of $\underline{\rho}_s$ welfare is given by

$$V(D_s, S_s, B_s, \underline{\rho}_s) = u(D_s + S_s). \quad (120)$$

If $\rho_s \rightarrow \underline{\rho}_s^-$, by definition of $\underline{\rho}_s$ welfare is given by

$$\lim_{\rho_s \rightarrow \underline{\rho}_s^-} V(D_s, S_s, B_s, \rho_s) = u(B_s + S_s). \quad (121)$$

As $\underline{\rho}_s$ is determined by

$$\frac{B_s + S_s}{D_s + S_s} = \phi \cdot \sum_{t=s}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \underline{\rho}_s)^{\frac{1-\nu^{t-s}}{1-\nu}} \cdot (1 + \bar{\rho})^{t-s - \frac{1-\nu^{t-s}}{1-\nu}}}, \quad (122)$$

the lemma is proven.

Proof of proposition 3. The proposition follows directly from proposition 2. We substitute ρ_D with the level prevailing along the good BGP.

Proof of corollary 3. Writing out the value for $\underline{\rho}^*$ of proposition 3, we have

$$1 - \phi - \phi \cdot \sum_{t=s+1}^{+\infty} \frac{(1 - \phi)^{t-s}}{(1 + \underline{\rho}^*)^{\frac{1-\nu^{t-s}}{1-\nu}} \cdot (1 + \bar{\rho})^{\frac{1-\nu^{t-s}}{1-\nu}}} = (1 - \phi) \cdot \frac{(1 + \bar{\rho})^{1-\alpha} - 1}{(1 + \bar{\rho})^{1-\alpha} - (1 - \phi)}. \quad (123)$$

Clearly, the solution for $\underline{\rho}^*$ is smaller than $\bar{\rho}$.

Proof of corollary 4. By total differentiation of equation (63) we find that

$$\frac{d\underline{\rho}^*}{d\nu} = - \frac{\frac{\partial \bar{F}(\underline{\rho}^*)}{\partial \nu}}{\frac{\partial \bar{F}(\underline{\rho}^*)}{\partial \rho}} \quad (124)$$

The derivative with respect to ρ is given by equation (59) and it is strictly positive for any value. The derivative with respect to ν is given by equation (60). Given that according to corollary 3 $\underline{\rho}^* < \rho$, evaluated at $\underline{\rho}^*$ the derivative is negative. Hence, the corollary is proven.