Online Appendix of "The Tipping Point: Interest Rates and Financial Stability"[a]

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A PROOFS

Proof of Proposition 1. Consider a subgame of the bank's problem starting at time $t \geq 1$. Using budget constraints (10) and (14), boundary condition (15) and anticipating equilibrium condition (7), I can write the intertemporal budget constraint

$$\phi \cdot \sum_{s=0}^{+\infty} \left(\frac{1-\phi}{1+\rho} \right)^{s} \cdot \prod_{j=0}^{s-1} (1 + r_{t+j}) \cdot (1-\phi)^{t-1} \cdot D_t \leq (1 + \delta \cdot q_t) \cdot B_t. \tag{OA.1}$$

Rearranging, I obtain condition

$$\phi \cdot \sum_{s=0}^{+\infty} \left(\frac{1-\phi}{1+\rho} \right)^{s} \cdot \prod_{j=0}^{s-1} (1 + r_{t+j}) \leq \frac{(1 + \delta \cdot q_t) \cdot B_t}{(1-\phi)^{t-1} \cdot D_t} \tag{OA.2}$$

that determines feasible deposit contracts $\left\{ r_{t+j} \right\}_{j=0}^{+\infty}$ conditional on initial values B_t and D_t and on price q_t.

First, I prove the "if" part of the proposition. For this, we verify that a deposit contract that satisfies incentive-compatibility constraints (9) and satisfies condition (OA.2), in

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which we impose initial conditions satisfying condition (17), exists. Notice that for a deposit contract that pays $r_{t+j} = 0$ for all $j \geq 0$, the left-hand side of equation (OA.2) is equal to $\phi \cdot (1+\rho)/(\phi+\rho)$. Moreover, the left-hand side of the equation is increasing in r_{t+j} and there is no upper bound on r_{t+j}. By this argument, the "if" part is proven.

Second, I prove the "only if" part of the proposition. This is equivalent to proving that the bank fails if condition (17) does not hold. The left-hand side of equation (OA.2) is equal to $\phi \cdot (1+\rho)/(\phi+\rho)$ when the incentive-compatibility constraint is binding at every date with $r_{t+j} = (1+\pi)^{-1} - 1$ for all $j \geq 0$. Since the left-hand side of equation (OA.2) is increasing in r_{t+j}, there is no deposit contract that is both incentive-compatible and feasible if condition (17) does not hold. Hence, the bank fails and the "only if" part of the proposition is proven. □

Proof of Proposition 2. Before I turn to the bank's problem, I look at the firm's problem to pin down equilibrium q_t^*. Arbitrage by the firm implies $1+\rho = (1+\delta \cdot q_{t+1}^*)/q_t^*$. The only sequence that satisfies this condition and no-bubble condition (6) is given by equation (7).

Now, I turn to the bank's problem. First, I solve the bank's full problem starting at time zero. Then, I verify that the solution is also a solution in every subgame of the problem. Using initial conditions (11) and (12), budget constraints (13) and (14), boundary condition (15), and the equilibrium condition (7), I can write the intertemporal budget constraint

$$\phi \cdot \sum_{t=1}^{+\infty} \left(\frac{1}{1+\rho}\right)^t \cdot (1-\phi)^{t-1} \cdot D_t = 1. \tag{OA.3}$$

Maximizing the objective function (8) subject to constraint (OA.3) with respect to choice variables $\{B_{t+1}, D_{t+1}\}_{t=0}^{+\infty}$, I obtain a set of optimality conditions. Once combined with (10), they can be written as

$$1 + r_0^* = (1+\rho)^\alpha \cdot \frac{(1+\rho)^{1-\alpha} - (1-\phi)}{\phi} \tag{OA.4}$$

and $1+r_t^* = (1+\rho)^\alpha$ for all $t \geq 1$. Notice that along the optimal path the incentive-compatibility constraints (9) are always slack. Hence, I can safely ignore them in this case. Re-arranging and combining initial conditions (11) and (12), budget constraints (13), (10) and (14), and the equilibrium condition (7), I can write a law of motion for the variable of interest given by

$$\frac{(1+\delta \cdot q_1) \cdot B_1}{(1-\phi)^0 \cdot D_1} = \frac{1+\rho}{1+r_0}, \tag{OA.5}$$

$$\frac{(1+\delta \cdot q_{t+1}) \cdot B_{t+1}}{(1-\phi)^t \cdot D_{t+1}} = \frac{1+\rho}{(1-\phi)\cdot(1+r_t)} \cdot \left[\frac{(1+\delta \cdot q_t)\cdot B_t}{(1-\phi)^{t-1} \cdot D_t} - \phi\right] \quad \text{for all } t \geq 1. \tag{OA.6}$$

Substituting in the optimal path $\{r_t^*\}_{t=0}^{+\infty}$, I confirm

$$\frac{(1+\delta \cdot q_t^*) \cdot B_t^*}{(1-\phi)^{t-1} \cdot D_t^*} = \frac{\phi \cdot (1+\rho)^{1-\alpha}}{(1+\rho)^{1-\alpha} - (1-\phi)} \quad \text{for all } t \geq 1. \tag{OA.7}$$

Finally, to verify that the solution above is also a solution to every subgame of the problem, take a subgame starting at time $t \geq 1$ with initial conditions B_t and D_t that satisfy condition (OA.7). The bank's optimality conditions imply that $1 + r_j^* = (1+\rho)^\alpha$ for all $j \geq t$. This proves the proposition. □

Proof of Corollary 1. Take the definition of the deposit franchise (21) and notice that

$$f\left(\{(1+\rho)(1+\pi)-1\}\right) = (1-\phi)\frac{(1+\rho)(1+\pi)-1}{(1+\rho)(1+\pi)-(1-\phi)}. \tag{OA.8}$$

With this, we can confirm that proposition 1 implies this corollary. □

Proof of Corollary 2. Applying the definitions of interest margin (20) and deposit franchise (21), we can confirm that proposition 2 implies the corollary. □

Proof of Proposition 3. Consider a subgame of the bank's problem starting at time $t \geq 1$. The interest rate is $\hat{\rho} > -(\phi+\pi)/(1+\pi)$. Using proposition 1, we can conclude that the bank does not fail as long as

$$\frac{(1+\delta\cdot\hat{q}_t)B_t}{(1-\phi)^{t-1}\cdot D_t}\geq\frac{\phi\cdot(1+\hat{\rho})\cdot(1+\pi)}{(1+\hat{\rho})\cdot(1+\pi)-(1-\phi)} \tag{OA.9}$$

with $\hat{q}_t = 1/(1+\hat{\rho}-\delta)$ for any values (B_t, D_t). Since the economy was running along its perfect-foresight equilibrium path before the time-t shock that changed the interest rate to $\hat{\rho}$, the initial conditions B_t and D_t satisfy

$$\frac{B_t}{(1-\phi)^{t-1}\cdot D_t}=\left[1-f_t\left(\{m_t^*\}\right)\right]\cdot\frac{1+\rho-\delta}{1+\rho}, \tag{OA.10}$$

as per corollary 2, where the perfect-foresight deposit franchise is given by equation (25). Subbing in these initial conditions, we can write the necessary and sufficient condition under which the bank does not fail as

$$\left[1-f_t\left(\{m_t^*\}\right)\right]\cdot\frac{1+\rho-\delta}{1+\rho}\geq\frac{\phi(1+\hat{\rho}-\delta)(1+\pi)}{(1+\hat{\rho})(1+\pi)-(1-\phi)}. \tag{OA.11}$$

Let us first study the values of $\hat{\rho}$ where such condition holds in the parameter region $\delta < 1-\phi$. The left-hand side of equation (OA.11) is not a function of $\hat{\rho}$ and is strictly larger than ϕ for these parameters. The right-hand side is continuous, tends to infinity for $\hat{\rho}\to-(\phi+\pi)/(1+\pi)^+$ and tends to ϕ for $\hat{\rho}\to+\infty$. By the intermediate value theorem, there is at least one point ρ^{tp} at which left-hand side and right-hand side are equal. Since the right-hand side is strictly decreasing in $\hat{\rho}$, ρ^{tp} is unique. To the left of ρ^{tp} the right-hand side is larger than the left-hand side of the equation. Hence, the bank fails. To the right of ρ^{tp}, the bank does not fail. Solving for $\hat{\rho}=\rho^{\mathrm{tp}}$ such that the left-hand side and the right-hand side are equal and substituting in equation (25) gives the tipping point

$$\left(1+\rho^{\mathrm{tp}}\right)(1+\pi)=1+m_t^*-\delta\cdot\frac{(\phi+m_t^*)\cdot\left[(1+\rho)(1+\pi)-(1+m_t^*)\right]}{(1-\phi)\cdot(1+\rho)-\delta\cdot(1+m_t^*)}. \tag{27}$$

Second, let us study the parameter region $\delta \in [1-\phi, 1)$. If I study the left-hand side of the inequality, I notice that it is larger than ϕ for $1-\phi \leq \delta \leq (1-\phi)\cdot(1+\rho)^{\alpha}$. It is strictly smaller than ϕ for $(1-\phi)\cdot(1+\rho)^{\alpha} < \delta < 1$. If I study the right-hand side for $\delta \geq 1-\phi$, I find that it is continuous, it tends to minus infinity for $\hat{\rho} \to -(\phi+\pi)/(1+\pi)^{+}$ and tends to ϕ^{-} for $\hat{\rho} \to +\infty$. Moreover, it is strictly monotonically increasing. This implies that the right-hand side is smaller than the left-hand side for any $\hat{\rho}$, whenever $1-\phi \leq \delta \leq (1-\phi)\cdot(1+\rho)^{\alpha}$. Hence, there is no shock such that the bank fails in this case. For $(1-\phi)\cdot(1+\rho)^{\alpha} < \delta < 1$, there is a unique ρ^{tp} at which right-hand side and left-hand side are equal. For $\hat{\rho} \leq \rho^{\mathrm{tp}}$, the left-hand side is larger than the right-hand side. Hence, the bank does not fail. For $\hat{\rho} > \rho^{\mathrm{tp}}$, the bank fails. Again, solving for $\hat{\rho} = \rho^{\mathrm{tp}}$ such that the left-hand side and the right-hand side are equal and substituting in equation (25) gives the tipping point (27). This concludes the proof. □

B ASSET LIQUIDATION

The assumption that the bank can sell any amount of bonds frictionlessly is innocuous for the paper's results. In this appendix, I confirm this.

At any given time t, the bank holds the equivalent of B_t new bonds.[1] Hence, it receives coupons amounting to B_t. Households withdraw in total $\phi \cdot (1-\phi)^{t-1} \cdot D_t$. The bank sells bonds if the coupon is insufficient to cover the withdrawals.

DEFINITION OA.1. *If at time t*

$$\frac{B_t}{(1-\phi)^{t-1} \cdot D_t} < \phi, \tag{OA.12}$$

a bank sells bonds.

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[1] The bank may not actually hold only new bonds but also older vintages of bonds. What matters is the new-bond-equivalent quantity of bonds it holds. For example, it may hold a bond issued at time $t-2$. This pays a coupon of δ at time t and is equivalent to δ new bonds issued at time $t-1$, as explained in section 3.

Whether a bank sells bonds is per se irrelevant for equilibrium outcomes in this paper's model, since bond selling is frictionless. Nonetheless, it is in principle interesting to introduce a liquidation cost in the bond market and study its effect on economic outcomes. This friction is theoretically compelling as a result of information asymmetries (Eisfeldt, 2004) and is emphasized in the literature on financial crises. For example, Diamond and Dybvig (1983) posit that assets have a higher per-period return if held for two periods rather than liquidated after one period, reflecting a liquidation cost.

I find that the introduction of a liquidation cost changes none of the results of this paper. Asset liquidation by the bank never happens, unless the bank is insolvent. In other words, the bank sells its bonds exclusively as a consequence of failure. It cannot become insolvent because of the poor terms at which it must sell bonds. Hence, the size of the shock that makes the bank insolvent does not depend on the liquidation cost.

PROPOSITION OA.1. *Consider an economy with $\delta \leq 1-\phi$. If in equilibrium the bank does not fail at time $t \geq 1$, then the bank does not sell bonds at any time $s \geq t$.*

Proof. By definition 1, if a bank does not fail at time $t \geq 1$, there exists a sequence $\left\{B_{j+1}, D_{j+1}, r_j\right\}_{j=t}^{+\infty}$ that, given initial conditions B_t and D_t, satisfies incentive-compatibility constraints (9), budget constraints (10) and (14), and the boundary condition (15). The subsequence $\left\{B_{j+1}, D_{j+1}, r_j\right\}_{j=s}^{+\infty}$ for $s \geq t$, given initial conditions B_s and D_s belonging to the above sequence, also satisfies incentive-compatibility constraints (9), budget constraints (10) and (14), and the boundary condition (15). Hence, there exists a solution to the subgame of the bank's problem starting at any time $s \geq t$. Again by definition 1, this implies the bank does not fail at any time $s \geq t$. Hence, by proposition 1 we have that the solvency condition

$$\frac{(1+\delta \cdot q_s) \cdot B_s}{(1-\phi)^{s-1} \cdot D_s} \geq \frac{\phi \cdot (1+\rho) \cdot (1+\pi)}{(1+\rho)(1+\pi)-(1-\phi)} \tag{OA.13}$$

holds for any $s \geq t$. Using equation (7) to substitute out q_s, we can verify that, under parametric condition $\delta \leq 1-\phi$, the solvency condition contradicts inequality (OA.12) in every period. □

Take the perfect-foresight equilibrium path described in proposition 2. This is the path the economy takes in equilibrium if it is never perturbed by a shock from time 0 on. Along this path the bank never fails and never sells any bonds. It meets withdrawals entirely with the coupons at every point in time. It follows that a liquidation cost would play no role in the economy's equilibrium outcomes. The above proposition is more general: even if an economy is hit by shocks, the bank never sells any bonds along the equilibrium path as long as it survives the shocks. In other words, a solvent bank never sells its bonds.

C PANICS

In this section, I introduce bank panics and show that they change neither the main theoretical results of the paper nor the quantitative results.

Panics are instances in which banks fail although they are solvent. They fail because so many households withdraw at a given point in time that the banks' resources are exhausted. Depending on the balance-sheet conditions of the bank, an incentive arises even for patient households to withdraw if everyone else does.

PROPOSITION OA.2. *At time $t \geq 1$, the equilibrium probability of bank failure is zero if condition*

$$(1+\delta \cdot q_t^*) \cdot B_t \geq (1-\phi)^{t-1} \cdot D_t \tag{OA.14}$$

holds. If solvency condition (17) does not hold, the equilibrium probability of bank failure is one. Otherwise, it is $\sigma \in [0,1]$.

Proof. Take the bank's intertemporal budget constraint at a given time $j > 0$ given by equation (16) and let all patient households withdraw at time j along with the impatient ones. This obtains condition $(1-\phi)^{j-1} \cdot D_j \leq (1+\delta \cdot q_j) \cdot B_j$. So, if and only if condition (OA.14) holds, then the intertemporal budget constraint holds regardless of how many households withdraw. The solvency condition remains the same as in the model without panics since the bank's budget constraints, from which it is derived, remain the same. If solvency condition (17) fails to hold, then the bank fails for sure. Since there is no element in the model to pin down the probability of a panic in the intermediate region between the solvency condition and condition (OA.14), I choose an exogenous parameter σ for this. □

Noticeably, the condition for the absence of bank failure is more stringent in the version of the model with panics. However, the condition below which banks are certain to fail remains the same.

It turns out that the bank chooses to be run-prone in the perfect-foresight equilibrium. This is a standard result in the literature, already found in Diamond and Dybvig (1983). It is optimal for banks to be run-prone, because this is the only way for banks to insure households against the risk of turning impatient early.

PROPOSITION OA.3. *For σ small enough, perfect-foresight equilibrium implies that*

$$1+r_t^* = [(1+\rho)\cdot(1-\sigma)]^{\alpha} \quad \text{for all } t \geq 1, \tag{OA.15}$$

$$(1+\delta \cdot q_t^*) \cdot B_t^* = \frac{\phi \cdot [(1+\rho)\cdot(1-\sigma)]^{1-\alpha}}{[(1+\rho)\cdot(1-\sigma)]^{1-\alpha} - (1-\phi)} \cdot (1-\phi)^{t-1} \cdot D_t^* \quad \text{for all } t \geq 1, \tag{OA.16}$$

q_t^* is given by (7) and the probability of bank failure is σ.

Proof. Let us guess and then verify that the bank chooses a deposit contract such that it is run-prone. Hence, the equilibrium probability of default at every date is σ. At a given time $t \geq 1$, default happens with probability $(1-\sigma)^{t-1} \cdot \sigma$ and the remaining $(1-\phi)^{t-1}$ households withdraw, receive their deposits D_t and store them until they turn impatient. With probability $(1-\sigma)^t$ the bank does not default and only the impatient households $(1-\phi)^{t-1} \cdot \phi$ withdraw at a given date $t \geq 1$. Thus, the households' time-0 expected utility is given by

$$[(1-\sigma) \cdot \phi + \sigma] \cdot \sum_{t=1}^{+\infty} [(1-\phi) \cdot (1-\sigma)]^{t-1} \cdot u(D_t). \tag{OA.17}$$

A bank that maximizes this objective function subject to incentive-compatibility constraints (9), initial conditions (11) and (12), budget constraints (10), (13) and (14), and boundary condition (15) sets r_t^* according to (OA.15). With this result, we can use the bank's budget constraints to confirm (OA.16). We can also check that the bank does not choose to be run-proof by comparing the time-zero expected utility of households under $r_t = \rho$ for all $t \geq 0$, which ensures the bank is run-proof, with the time-zero expected utility of households under the deposit contract described in the proposition. For $\sigma \to 0$, it is easy to confirm that this is the case. □

The presence of bank panics changes the deposit contract offered by banks in equilibrium. In particular, the bank offers a lower deposit rate. The bank has an incentive to increase the extent to which it provides liquidity-risk insurance to ensure that, if a panic takes place early, depositors get a high payout. The flip side of this is that the bank must earn a larger interest margin over time in order to be solvent.

The key theoretical results of the paper, contained in proposition 3, remain the same *conditional on the perfect-foresight interest margin*. In fact, the proof of proposition 3 makes use of the bank's solvency condition, which is unchanged with panics as shown in proposition OA.2, and is valid for general initial conditions (B_t^*, D_t^*) and a general perfect-foresight interest margin m_t^*.

If one knows the perfect-foresight interest margin, one can quantify the tipping point regardless of the probability of bank panics. This is important for the quantitative results in section 7 because banks' interest margins are indeed observable. As it turns out, the quantitative exercise remains almost entirely valid with bank panics. The only parameter that needs a new calibration is the coefficient of relative risk aversion $1/\alpha$. This is unimportant because the parameter does not appear either in the condition on the dominance of the effects or in the tipping point formula. Nevertheless, I find that a model with panics needs a lower degree of relative risk aversion to generate the same average interest rate and deposit rate.

D RISK AND ENDOGENOUS BOND DURATION

In this section, I analyze the bank's endogenous choice of variable δ, which characterizes the duration of bonds. In the main text, banks optimize under certainty. Hence, they are indifferent between different levels of δ and I can treat δ as a parameter in the analysis. However, it is interesting to study the bank's ex-ante behaviour if it anticipates that the environment is risky.

I introduce risk in the rate of return on bonds. Before a given time $s \geq 1$, the return is ρ with certainty. At time $s \geq 1$, the return $\hat{\rho}_s$ is random. The mean rate of return is ρ. The variance of the shock is small with $\sigma_\rho^2 \to 0$.[2] Also, the shock is permanent, meaning that $\hat{\rho}_{s+j} = \hat{\rho}_s$ for all $j \geq 0$. Additionally, I introduce risk in the coupon paid by bonds. At time $s \geq 1$, a bond issued at time $t-k$ pays a coupon $\zeta_s \delta^{k-1}$. On average, $\zeta_s = 1$. Again, the variance is small with $\sigma_\zeta^2 \to 0$ and the shock is permanent with $\zeta_{s+j} = \zeta_s$ for all $j \geq 0$. The coupon shock is also independent of the shock to the rate of return. We can think of coupon risk as counterparty risk. This setup implies that we can use

$$q_t^* = \frac{\zeta_t}{1 + \hat{\rho}_t - \delta} \tag{OA.18}$$

instead of equation (7) for any time period $t \geq s$.[3]

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[2] Small risk is adopted for tractability and is enough to pin down endogenous bond duration.

[3] Equation (7) holds in expectations before the realization of the shock at time s, so that $\mathbb{E}_t\left(q_{s+j}^*\right) = (1+\rho-\delta)^{-1}$ for all $t < s$ and $j \geq 0$.

Given that both shocks realize at time s, the time-s subgame to the bank's problem is deterministic. This implies that, conditional on the shock realization, the initial conditions B_s and D_s, and bond price q_s^*, it has the perfect-foresight solution contained in proposition 2. The bank chooses $1 + r_{s+j}^* = (1 + \hat{\rho}_s)^\alpha$ for all $j \geq 1$. Combining this with expected utility (8) and the law of motion of deposits (10), we can write the expected utility of a patient household at time s as[4]

$$V_s = \frac{\alpha}{1-\alpha} \left\{ 1 - \frac{\phi\,(1 + \hat{\rho}_s)^{1-\alpha}}{\left[(1 + r_s^*)\,D_s\right]^{\frac{1-\alpha}{\alpha}} \left[(1 + \hat{\rho}_s)^{1-\alpha} - (1 - \phi)\right]} \right\}. \tag{OA.19}$$

Initial conditions B_s and D_s, and the bond price q_s^* have an impact on the households' utility via the time-s deposit rate, given by

$$1 + r_s^* = (1 + \hat{\rho}_s)^\alpha \, \frac{(1 + \hat{\rho}_s)^{1-\alpha} - (1 - \phi)}{\phi(1 - \phi)} \left[\frac{(1 + \delta q_s^*) B_s}{(1 - \phi)^{s-1} D_s} - \phi \right] \tag{OA.20}$$

as implied by the intertemporal budget constraint.[5]

In principle, the fact that there is risk at time $s \geq 1$ gives the bank a precautionary motive. In anticipation of the shock, the bank reduces deposit rates and builds up a buffer of assets to support households' consumption in case of an adverse shock at time s. However, because the variance of shocks tends to zero, the deposit-rate choice is equal to the perfect-foresight choice. This implies that the economy's state variables follow the perfect-foresight path as described in proposition 2, which implies that B_s and D_s comply with

$$\frac{B_s}{(1 - \phi)^{s-1} D_s} = \frac{\phi(1 + \rho - \delta)}{(1 + \rho)^\alpha \left[(1 + \rho)^{1-\alpha} - (1 - \phi)\right]}. \tag{OA.21}$$

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[4] Notice that constant relative risk aversion implies that the functional form of the felicity function is $u(C) = \left(C^{1-\frac{1}{\alpha}} - 1\right) \big/ \left(1 - \frac{1}{\alpha}\right)$.

[5] The derivation of r_s is equivalent to the derivation of r_0^* in the proof of proposition 2.

Plugging this result into the expression for the time-s deposit rate (OA.20), we obtain that

$$1+r_s^*=\left(\frac{1+\hat{\rho}_s}{1+\rho}\right)^{\alpha}\frac{(1+\delta q_s^*)(1+\rho-\delta)-\left[(1+\rho)^{1-\alpha}-(1-\phi)\right](1+\rho)^{\alpha}}{1-\phi} \tag{OA.22}$$

as risk tends to zero as assumed in this section.[6] Finally, we can plug this into utility (OA.19) to obtain households' time-s utility conditional on the shocks:

$$\begin{aligned}&\frac{1-\alpha}{\alpha}V_s-1=\\&=-\frac{\phi(1+\rho)^{1-\alpha}(1-\phi)^{\frac{1-\alpha}{\alpha}}}{D_s^{\frac{1-\alpha}{\alpha}}\left[(1+\hat{\rho}_s)^{1-\alpha}-(1-\phi)\right]\left\{\frac{1+\hat{\rho}_s+(\zeta_s-1)\delta}{1+\hat{\rho}_s-\delta}(1+\rho-\delta)-(1+\rho)^{\alpha}\left[(1+\rho)^{1-\alpha}-(1-\phi)\right]\right\}^{\frac{1-\alpha}{\alpha}}}.\end{aligned} \tag{OA.23}$$

With a Taylor expansion around the means of the shocks, we can write the exposure of utility to shocks as

$$\begin{aligned}\mathrm{d}V_s=&\frac{\phi}{D_s^{\frac{1-\alpha}{\alpha}}\left[(1+\rho)^{1-\alpha}-(1-\phi)\right](1-\phi)(1+\rho)^{\alpha}}\cdot\\&\cdot\left\{\delta\mathrm{d}\zeta_s+\left[\frac{1-\phi}{\alpha\left[(1+\rho)^{1-\alpha}-(1-\phi)\right]}-\frac{\delta}{1+\rho-\delta}\right]\mathrm{d}\hat{\rho}_s\right\}+o\left(\mathrm{d}\hat{\rho}_s+\mathrm{d}\zeta_s\right).\end{aligned} \tag{OA.24}$$

Shocks to coupons only impact bond prices. Hence, they matter for household utility only insofar as the bank is exposed to bond prices by having a strictly positive δ. Positive shocks to returns increase utility for given bond prices. However, they also reduce bond prices and this has an adverse effect on utility. The latter effect is an increasing function of δ because this parameter captures the bank's exposure to bond prices.

At the limit for infinitesimally small shocks, the remainder $o\left(\mathrm{d}\hat{\rho}_s+\mathrm{d}\zeta_s\right)$ drops out

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[6] Note that on average the time-s deposit rate is on the perfect-foresight path with $\mathbb{E}_t\left(1+r_s^*\right)=(1+\rho)^{\alpha}$ for all $t<s$.

and we can work out the variance of household utility:

$$\frac{\mathbb{E}_0\,(\mathrm{d}V_s)^2}{\sigma_\zeta^2} = \left\{ \frac{\phi}{D_s^{\frac{1-\alpha}{\alpha}}\left[(1+\rho)^{1-\alpha} - (1-\phi)\right](1-\phi)(1+\rho)^{\alpha}} \right\}^2 \cdot \\ \cdot \left\{ \delta^2 + \left[\frac{1-\phi}{\alpha\left[(1+\rho)^{1-\alpha} - (1-\phi)\right]} - \frac{\delta}{1+\rho-\delta} \right]^2 \frac{\sigma_\rho^2}{\sigma_\zeta^2} \right\}. \quad \text{(OA.25)}$$

The equilibrium deposit contract, which maximizes households' utility as of time zero, sets δ to minimize the variance of time-s utility.[7] This is optimal as of time zero because the choice of δ has no impact on average utility as can be checked in equation (OA.23).[8] Also, because utility is concave, expected utility is decreasing in its volatility. Hence, the endogenous bond duration is given by

$$\delta^* = \left\{ \frac{1-\phi}{\alpha\left[(1+\rho)^{1-\alpha} - (1-\phi)\right]} - \frac{\delta^*}{1+\rho-\delta^*} \right\} \frac{1+\rho}{(1+\rho-\delta^*)^2} \frac{\sigma_\rho^2}{\sigma_\zeta^2} \quad \text{(OA.26)}$$

If shocks to returns are prevalent (i.e., $\sigma_\rho^2 \big/ \sigma_\zeta^2 \to +\infty$), then the bank chooses to set

$$\delta^* = \frac{(1-\phi)(1+\rho)}{\alpha(1+\rho)^{1-\alpha} + (1-\alpha)(1-\phi)} > (1-\phi)(1+\rho)^{1-\alpha}, \quad \text{(OA.27)}$$

which, as according to proposition 3, implies the existence of a tipping point such that the bank is vulnerable to increases in the interest rate. Because reductions in the return on bank assets reduce household consumption, long-term assets provide valuable insurance. On the other hand, if coupon shocks are prevalent, then it is optimal for the bank to set $\delta^* = 0$. This implies the existence of a tipping point such that the bank is vulnerable to reductions in the interest rate in this case.

With endogenous bond duration, the tipping point's level and direction depend on the mix of shocks faced by the bank as described in equation (OA.26). Calibrating with

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[7] The firm is indifferent about δ because it is risk-neutral.

[8] This is why under perfect foresight an optimal δ is not determined.

the values used in section 7 and assuming that shocks to returns are the only source of risk, we obtain endogenous duration of $\delta^*/(1+\rho-\delta^*) = 85.29$ years. In practice, we do not observe such extreme duration in banks' asset portfolios. The model can accommodate optimally shorter bonds, all the way to an endogenous duration of 0 years, with the introduction of counterparty risk with $\sigma_\zeta^2 \big/ \sigma_\rho^2 > 0$.

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